# Optimal Scheduling and Risk Minimization for a GenCo under Market Conditions

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ABSTRACT

*In this paper, an alternative formulation for the optimal scheduling of a generator is presented. Here, instead of using the traditional formulation of maximizing the expected profits (where a generator is risk-neutral), we model the problem with the supplier being risk-averter. The problem is solved by finding the schedule that produces the least risk for a given minimum target of expected profits. The variance in prices is used to consider the risk in the generation schedule. Usually, scheduling and risk management are separately done. On one hand, scheduling is generally related to maximize the expected profits while , on the other hand, risk is used mainly to analyze financial positions. In this paper, we use a model that implicitly consider both task; it is done by introducing risk into the objective function of the stochastic problem by means of the Markowitz model. Results are presented for a 24-hour trading period.*

## 1. Introduction

Traditionally, Electric Power Systems (EPSs) were vertically constituted. Generation, transmission and distribution were controlled (or even owned) by a unique entity. Since 1982, when Chile restructured its EPS, a deep de-regulation trend has taken place around the world. The new era of managing EPS is based on introducing competition in generation, transmission and distribution. Due to the economical impact, the concern has been mainly focused in generation and transmission.

In the traditional vertically integrated scheme, a central entity dispatches all the generation units in order to achieve the minimum-cost operation, taking into account the reliability and security requirements of system. This entity, knowing the characteristics of the units, determines which plants must to start (and also at which level of production to run), and which must shut down. This problem is called *unit commitment* (UC). In the traditional UC, the demand is forecasted (say, day ahead), while all the system elements are centrally operated.

However, in some electricity markets, a centralized UC is not implemented anymore. In the market environment, generation units are operated by private companies; these *utilities* sell energy – based on bids – in a Power Market. Then the bids that meet the optimal operation are chosen. In the case of competitive markets, it is well known that the best strategy for a utility to maximize profit – due to energy sells – is to bid its marginal cost. However, there are other costs and factors to be considered in order to have the optimal strategy to sell energy. Typical models for scheduling are based on the maximization of expected profit, considering that the supplier is risk-neutral. The main drawback of this approach is that they do not reflect the risk of decisions. Usually, scheduling and risk management are done by separate studies. In this paper, we use a model that implicitly consider both task; it is done by introducing risk into the objective function of the profit-maximization problem by means of the Markowitz model.

This paper is organized as follows. In Section II, the scheduling problem for a generator is presented. A continuous approach to the scheduling problem is develop in Section III, while a Mixed-Integer formulation is given in Section IV. In section V, some issues about profit anfd risk in scheduling are described. Conclusion in Section VI closes the paper.

## 2. Problem Formulation

Since a utility has to submit simple energy bids (price-quantity) to the market. It, by its owns, has to internalize in the bid a set of costs and operational constraints in order to define its optimal sschedule to sell energy in such a format through a horizon plan, (say, a day horizon in hourly periods). Some facts to be considered in the UC problem (for a thermal unit) are [1].

- Variable cost. They represent the production cost due to fuel.
- Maximum and minimum generation levels.
- Start up costs. This is a common restriction for thermal plants; the reason is that an amount of energy is needed to bring the unit online. For simplicity, this cost can be modelled simply as a fixed value for starting the generation unit.
- Minimum up/down time. These constraints are imposed by the physical operation of the plant (the thermal cycle); that is, once the unit is off, there is a minimum time required to be online. Similarly, once the unit is on, there is a minimum time to shut down the unit. A common way to consider these constraint is by using a maximum number of (times) startups and shutdowns for the unit.
- Up- and Down-ramp rates. A thermal unit can not increase/decrease its output at any level; there are maximum and minimum rates. These constraints couple together

every current period of the analysis with the previous and the following one.

In addition, a utility has to consider expected market conditions to define its schedule. Since a generation unit is placed in a node of the EPS, and the nodal price is a by-product of the whole system – and market – condition such as changing (uncertain) demand and congestion, the utility faces an uncertain price to define its schedule because of exogenous factors. The objective of a utility is to maximize the expected profits – revenues minus costs –. Therefore, the UC problem can be modelled as a profit maximization one under uncertainty to sell energy.

Let us denote as $\mathcal{T}$ the set of hourly trading periods, $t$, for the planning horizon; $p_t$ and $\rho_t$ denote the generation power and nodal price, at period t, respectively; $ct_t$ stands for the total costs – variable and fixed – at period $t$, i.e.,

$$ct_t = cv_t(p_t) + \zeta\, z_t + \xi\, w_t \tag{1}$$

where $\zeta$ and $\xi$ denote the startup and shutdown costs, respectively; $z_t, w_t \in \{0,1\}$ denote the integer variables which defines if the generation unit has been put off- or on-line. The generation (variable) costs are represented by a nondecreasing convex cost function, i.e.,

$$cv_t = \gamma p_t^2 + \beta p_t \tag{2}$$

The profit for the generator in the period $t$ is given by

$$\Pi_t = \rho_t p_t - ct_t(p_t), \tag{3}$$

Thus, the profit maximization problem can be mathematically stated as follows:

$$\begin{aligned}
max \quad & \sum_t E\big[\Pi_t\big] & (4)\\
s.t. \quad & u_t p_t - u_{t+1}p_{t+1} \le \underline{\triangle} p\,, \quad t < |\mathcal{T}| & (5)\\
& u_{t+1}p_{t+1} - u_t p_t \le \overline{\triangle}_p\,, \quad t < |\mathcal{T}| & (6)\\
& u_t \underline{p}_t \le p_t \le\ u_t \overline{p_t}\,, \quad \forall t \in \mathcal{T} & (7)\\
& \sum_t z_t \le \tau_{on} & (8)\\
& \sum_t w_t \le \tau_{off} & (9)\\
& z_t, w_t, u_t \in \{0,1\}\,, \quad \forall t \in \mathcal{T} & (10)
\end{aligned}$$

where $E[\cdot]$ stands for the expected profit; $\overline{\triangle p}_{gl}$ and $\underline{\triangle p}_{gl}$ are the up- and down-ramp rates, respectively. Equation (5) and (6) define the ramp rates constraints, while Equation (7) stands for the max/min generation output constraint. Equation(8) and (9) are simplified expressions to include the minimum up/down time constraints; $\tau_{on}$ and $\tau_{off}$ are the maximum number of times to start up and shut down the unit along the planning horizon.

This maximization problem of expected profits problem is a stochastic, mixed-integer, and non-convex, problem.

## 3. A Continuous Approach

The profit maximization problem is firstly solved by relaxing the integrality requirement; neglecting variables $z_t, w_t$ and $u_t$ implies that the generator can startup and shutdown whenever it is required; obviously, it is far from real conditions. However, this relaxation can be also seen from a different point of view. With risky bids, the market outcome can produce infeasible conditions for a generator such that its operational requirements (start up and shutdown) would be violated if it operates, the generator needs a hedge for these potential outcomes. The simplest way for hedging, in this case, is to sign a bilateral contract with a load such that, at least, the minimum generation is engaged; hence, the generator must be on-line for the entire planning horizon, and the integrality requirements of the problem can be ignored. Thus, the profit maximization problem is now a stochastic convex problem.

### *A.* Deterministic Problem

The simplest procedure to start is to neglect the market uncertainty in the model, it can be done by taking a historical set of prices, for instance, the historical set of the same previous day, the last set of market prices or any other set of prices. However, these approaches may be flawed since they can produce a sub- or over-estimation of the profits with no knowledge of the risk of such a schedule. Another practical alternative is to forecast the next-day prices (*i.e,* with times series) [2]. Here, the expected set of prices, based on historical data, is considered to implement the Deterministic case. As the set of prices is given, the problem is reduced to a quadratic programming one. It is mathematically stated as follows:

$$\begin{aligned}
max \quad & \sum_t \left\{\rho_t p_t - \gamma p_t^2 - \beta p_t\right\} & (11)\\
s.t. \quad & p_t - p_{t+1} \le \overline{\triangle p}\,, \quad t = 1,2,...,|\mathcal{T}|-1 & (12)\\
& p_{t+1} - p_t \le \underline{\triangle p}\,, \quad t = 1,2,...,|\mathcal{T}|-1 & (13)\\
& \underline{p}_t \le p_t \le \overline{p_t}\,, \quad \forall t \in \mathcal{T} & (14)
\end{aligned}$$

By using matrix notation and introducing slack variables, this problem can be fairly written as a convex programming problem, *i.e.,*

$$\begin{aligned}
max \quad & \mathbf{x}^T\mathbf{Q}\ \mathbf{x} + \mathbf{d}^T\mathbf{x} & (15)\\
s.t. \quad & \mathbf{Ax} = \mathbf{b} & (16)\\
& \mathbf{x} \ge \mathbf{0} & (17)
\end{aligned}$$

where $\mathbf{Q}$ is a diagonal matrix which contains the coefficients $\gamma$; the vector $\mathbf{d}$ has as elements the differences $\rho_t - \beta$; the matrix $\mathbf{A}$ is conformed by the linear constraints of the problem; the vector $\mathbf{b}$ contains the maximum and minimum limits of operation and $\mathbf{x} \in \mathcal{R}^{|\mathcal{N}|}$ is the vector of primal variables which is conformed by the generation and the slack variables. This problem has been solved by using a primal-dual interior-point method, the description is given in Appendix A. In this simulation, it is assumed linear and quadratic cost of $\$10/MW$ and $\$0.2/MW^2$, while both Up- and Down-ramp rate limits are $\$10/MW$. The outcome for the deterministic

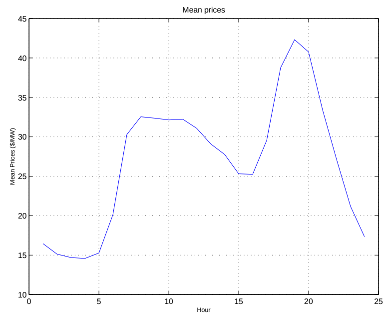


Fig. 1. Prices for the Deterministic case

case is shown in Fig. 2; by comparing the generation schedule with the prices pattern (Figure 1), it can be seen that the generation schedule follows the trend of the forecasted prices. Daily profit is shown in Table I.

TABLE I
DETERMINISTIC CASE

| Total Revenue | Total Cost | Total Profit |
|---|---|---|
| \$30613 | \$20165 | \$10448 |

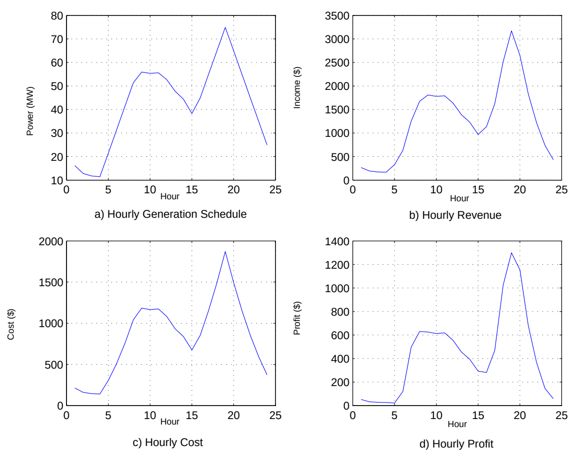


Fig. 2. Deterministic Solution

## B. STOCHASTIC PROBLEM

The stochastic problem of profit maximization is solved by using the minimization of the variance; in this approach, the objective is to minimize the uncertainty of energy sales given a target value of expected profits, *i.e.,* it determines the schedule by finding the combination of hourly generations which gives the lowest risk for the given profit; this formulation trades off between the expected profits and the risk taken. This approach, developed by Markowitz, relies in two assumptions [3]: i) the expected profit is normally distributed; and ii) the supplier is risk-averter and prefers lower risks. We also assume that a current hourly price is correlated only with the previous and next one; hence, the structure of the covariance matrix is diagonal-like.

Thus, the minimization of the risk is mathematically stated as follows:

$$min \quad \mathbf{x}^T \mathbf{H}\ \mathbf{x} \tag{18}$$

$$s.t. \quad \mathbf{x}^T \mathbf{Q}\ \mathbf{x} + \mathbf{d}^T \mathbf{x} \geq L \tag{19}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{20}$$

$$\mathbf{x} \geq \mathbf{0} \tag{21}$$

where $\mathbf{H}$ is the variance matrix for the prices, and Equation (19) stands for the target value, $L$, of expected profit. Since the constraint (19) is convex, the variance minimization problem is still a convex problem. The implementation of this approach is an extension from the deterministic case (see Appendix A). Simulations for different target values have been tested; the money stream for a target value of $\$7000$ is given in Table II, and the solution computed is shown in Figure 3. In this case, it is interesting to notice that the pattern of the daily generation schedule does not follow the pattern of the expected prices on peak hours as it has higher variance. Finally, a comparison of schedules for different target value is shown in Figure 4.

TABLE II
STOCHASTIC CASE

| Expected Income | Expected Cost | Expected Profit | Standard Dev. |
|---|---|---|---|
| \$ 14082 | \$7082 | \$7000 | 1278 |

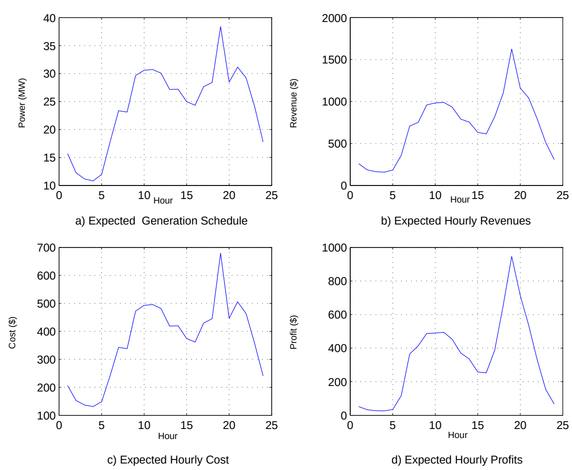


Fig. 3. Stochastic case with target value of \$7000

# 4. A MIXED-INTEGER MODEL

The original stochastic, quadratic and mixed-integer problem, is solved by using the *Mixed Integer Nonlinear Programming (MINLP)* subroutine. In this simulation, it is assumed the generator is initially off; a comparison between the solutions of the continuous case, implemented with the primal-dual method, and the mixed-integer one, shows that both produce the same generation schedule with an exception in period one where the mixed-integer problem has to start from a level of zero –see Fig. Figure 5. As the generator has to be within the ramp rate limits, the only feasible level for the generator is

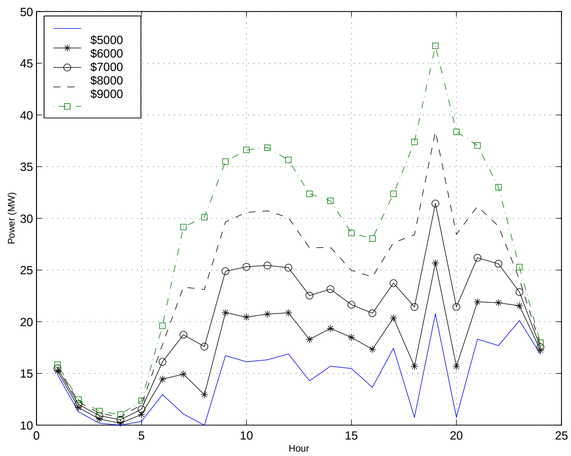


Fig. 4. Comparison of generation schedule for different target values

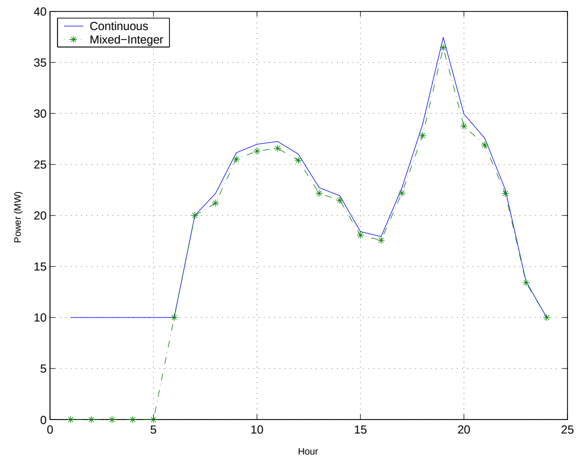


Fig. 6. Stochastic case

$10MW$. It is no surprising that both solutions coincides; since the generation cost are cheap enough, the generation schedules computed are from the minimum level. By increasing the

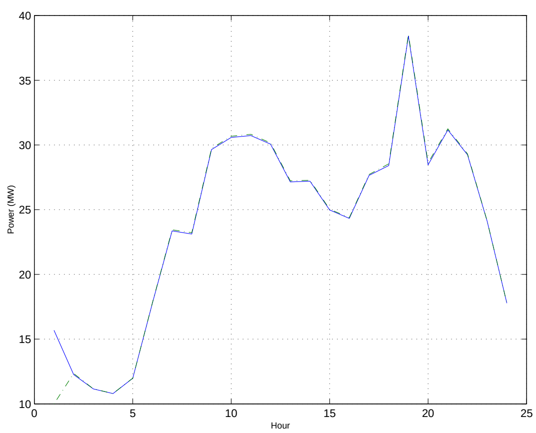


Fig. 5. Stochastic case. Cheap Generation Costs

generation cost and reducing the start up and shut down cost, we force a more critical conditions so that we can see the incidence of having the integrality conditions, as shown in Figure 6. In the continuous case, because of the generation costs, the generator is scheduled at its minimum level in the first five periods; while for the mixed-integer case, it is cheaper to have it off. However, both schedules are computed for the same target value of $\$5000$. For the remaining periods, the schedule is quite similar.

## 5. On the Risk of Scheduling

As mentioned above, the main feature of the mean-variance approach is that it lets the tradeoff between profit and risk be identified by using the standard deviation (variance). For instance, consider the generation schedules $A, B$ and $C$ produced with target values of $\$4000, \$7000$ and $\$1000$, (from Section 3-B) respectively. Their Probability Distribution and Cumulative functions are plotted in Figure 7. It can be seen that the higher the target value, the higher the standard deviation, the broader the probability distribution, and the greater the risk. From the Cumulative distribution, it is unlikely that schedule $C$ has profits below $\$1500$. In addition, the probability that the schedules $A, B$ and $C$ have a profit greater than, say, 6000, is $.06\%$, $78.3\%$ and $95\%$, respectively. Afterwards, with simulations for different target values, a set

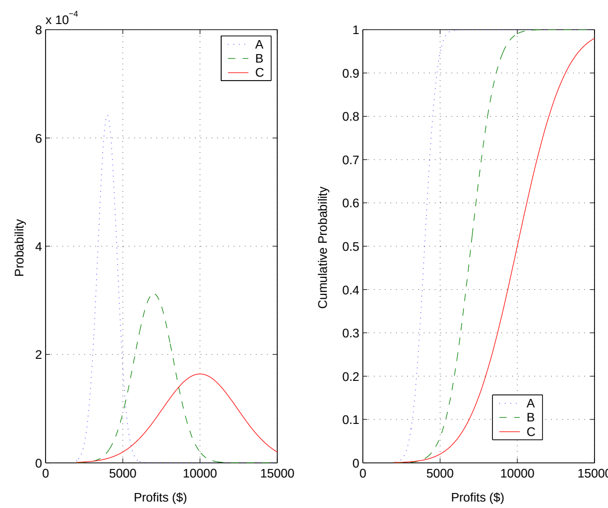


Fig. 7. Normal Probability and Cumulative Distribution Functions

of scenarios are produced; they are depiected in Figure 8; this curve represents all the possible generation schedules; that is, it comprises all the possible combinations of expected profit *vs.* risk where for a given expected profit no other schedule with lower risk can be found; this basically is the definition of the *Efficient Set* [4], known also as the *Efficient Frontier* [5]. In our case, the efficient set coincides with the feasible set; *i.e.*, the risk grows as the expected profit increases through the whole set.

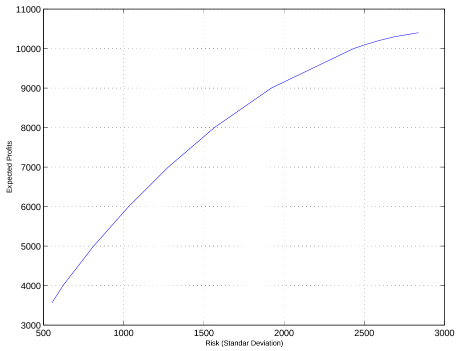


Fig. 8. Efficient Set for Generation Schedules

## 6. Conclusion

In this paper, an alternative formulation for the maximization of profits for a energy supplier has been addressed. Here, instead of using the traditional formulation of maximizing the expected profits, where the supplier is risk-neutral; we model the problem where the supplier is considered risk-averter. The problem is solved by finding the schedule that produces the least risk for a given minimum target of expected profits. Nonetheless, a potential drawback of this model is the assumption of having a normally distributed function of profit. The variance is used as a tool to consider the risk in the generation schedule. From the simulations, it is seen that the deterministic case produces an schedule too optimistic compared to the stochastic case. The upward curve of the efficient set denotes that expecting higher profits requires to assume higher risk.

## Acknowledgment

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## Appendix A
## Solution by an Interior Point Method

By introducing a nonnegative slack variable in the expected profit (Equation 19), the model (18)-(21) becomes

$$\begin{aligned} min \quad & \mathbf{x}^T\mathbf{H}\ \mathbf{x} && (22)\\ s.t. \quad & \mathbf{x}^T\mathbf{Q}\ \mathbf{x} + \mathbf{d}^T\mathbf{x} - y - L = 0 && (23)\\ & \mathbf{Ax} - \mathbf{b} = 0 && (24)\\ & \mathbf{x}, y \geq \mathbf{0} && (25) \end{aligned}$$

Its associated Lagrangian function, $\pounds$, can be defined as

$$\pounds(y, \boldsymbol{x}, \lambda_1, \boldsymbol{\lambda}_2) = \mathbf{x}^T\mathbf{Hx} - \lambda_1(\mathbf{x}^T\mathbf{Qx} + \mathbf{d}^T\mathbf{x} - y - L) - \boldsymbol{\lambda}_2^T(\mathbf{Ax} - \mathbf{b}) - s_1 y - \mathbf{s}_2^T\mathbf{x} \qquad (26)$$

where $\lambda_1$ and $\boldsymbol{\lambda}_2$ are the unrestricted dual variables – Lagrange multipliers – associated to the expected profit and operational constraints, respectively; $s_1 \geq 0$ and $\mathbf{s_2} \geq 0$ are the dual variables associated to the primal variables. The Karush-Kuhn-Tucker – first order necessary conditions – for optimality can be casted as a equations system, *i.e.,*

$$\begin{aligned} \nabla_{\mathbf{x}}\mathcal{L} = &\ -2\mathbf{Hx} + \lambda_1(2\mathbf{Qx} + \mathbf{d}) + \mathbf{A}^T\lambda_2 + s_2 && = 0\\ \nabla_y\mathcal{L} = &\ -\lambda_1 + s_1 && = 0\\ \nabla_{\lambda_1} = &\ \mathbf{x}^T\mathbf{Qx} + \mathbf{d}^T\mathbf{x} - y - L && = 0\\ \nabla_{\boldsymbol{\lambda}_2} = &\ \mathbf{Ax} - \mathbf{b} && = 0 \end{aligned}$$

with the complementary conditions

$$\begin{aligned} s_1 y &= 0 && (27)\\ s_{2i}x_i &= 0, \quad \forall i \in \mathcal{N} && (28) \end{aligned}$$

Let the KKT's conditions be rewritten as a system of non-linear equations given by

$$\boldsymbol{F}(\boldsymbol{v}) = \begin{pmatrix} -2\mathbf{Hx}_1 + \lambda_1(2\mathbf{Qx} + \mathbf{d}) + \mathbf{A}^T\boldsymbol{\lambda}_2 + \boldsymbol{s}_2\\ -\lambda_1 + s_1\\ \mathbf{x}^T\mathbf{Qx} + \mathbf{d}^T\mathbf{x} - y - L\\ \mathbf{Ax} - \mathbf{b}\\ s_1 y\\ \boldsymbol{S}_2\mathbf{X} \end{pmatrix} = 0$$

where $\boldsymbol{S}_2 := diag(s_{2i})$, $\boldsymbol{X}_2 := diag(s_{2i})$ and $\mathbf{v} = (y, \boldsymbol{x}, \lambda_1, \boldsymbol{\lambda_2}, s_1, \boldsymbol{s_2})$ is the vector of primal and dual variables.

By taking the terms of the first- order Taylor expansion, the problem (22)-(25) can be solved, for a current point $\boldsymbol{v}^k$, by the so-called Newton method, *i.e.,*

$$\boldsymbol{v}^{k+1} = \boldsymbol{v}^k - \alpha^k[\nabla_v\boldsymbol{F}(\boldsymbol{v})]^{-1}F(\boldsymbol{v}^{\boldsymbol{k}}), \qquad \alpha^k \in (0, 1] \qquad (29)$$

Here, the Jacobian, $\nabla_v\boldsymbol{F}(\boldsymbol{v})$, is given by

$$\nabla_v\boldsymbol{F}(\boldsymbol{v}) = \begin{pmatrix} -2\mathbf{Hx} + \lambda_1\mathbf{Q} & 0 & 2\mathbf{Qx} + \mathbf{d} & A^T & 0 & \mathbf{I}\\ 0 & 0 & -1 & 0 & 1 & 0\\ (2\mathbf{Qx} + \mathbf{d})^T & -1 & 0 & 0 & 0 & 0\\ A & 0 & 0 & 0 & 0 & 0\\ 0 & s_1 & 0 & 0 & y & 0\\ S_2 & 0 & 0 & 0 & 0 & \mathbf{X} \end{pmatrix}$$

while $[\nabla_v F(v)]^{-1}F(v^k)$ is the search direction identified simply as $\Delta v^{af}$. However, due to the nonnegative constraints, usually a full Newton step $(\alpha = 1)$ can not be directly taken; but, even if feasible steps are taken, the convergence of the method would be quite poor [6]. In order to improve convergence, a variety of interior-point methods has been generated, being one of the most efficient the second order predictor-corrector proposed by Methrotra [7]. In order to efficiently solve the profit maximization problem, the so-called primal-dual interior-point method has been extended and implemented as follows.

The Methrotra algorithm proposes to construct the search direction $\Delta v$ by means of three components

$$\Delta \boldsymbol{v} = \Delta \boldsymbol{v}^{af} + \Delta \boldsymbol{v}^{ce} + \Delta \boldsymbol{v}^{co} \tag{30}$$

where the superindices $(^{af}), (^{ce}), (^{co})$ stand for the predictor, centering and corrector directions, respectively.

*Predictor component*. This component makes progress towards the optimal solution; it is a linear approximation of (22)-(25) and is computed from

$$\nabla_v \boldsymbol{F}(\boldsymbol{v}) \Delta \boldsymbol{v}^{af} = -\boldsymbol{F}(\boldsymbol{v}) \tag{31}$$

In order to satisfy the nonnegativity constraints, maximum stepsizes for the primal and dual variables are computed, *i.e.,*

$$\alpha^p_{pred} = min\Big\{1, \min_{y^{pred}, x_i^{pred} < 0} \Big( -\frac{y}{\Delta y^{pred}}, -\frac{x_i}{\Delta x_i^{pred}} \Big)\Big\} \tag{32}$$

$$\alpha^d_{pred} = min\Big\{1, \min_{s_1^{pred}, s_{2_i}^{pred} < 0} \Big( -\frac{s_1}{\Delta s_1^{pred}}, -\frac{s_{2i}}{\Delta s_{2_i}^{pred}} \Big)\Big\} \tag{33}$$

Unlike the linear case, in the formulation for the convex programming problem, the primal and dual variables are coupled through the quadratic terms; therefore, if separate stepsizes are taken for primal and dual variables, it may break down the feasibility. Hence, a sole stepsize is used for both set of variables, *i.e,*

$$\alpha^{pred} = min\{\alpha^p_{pred}, \alpha^d_{pred}\} \tag{34}$$

*Centering component*. It bias the current direction toward the interior of the feasible region. It is done by using a centering parameter define by

$$\sigma = \Big(\frac{\mu^{pred}}{\mu}\Big)^3 \tag{35}$$

where $\mu$ is the current complementary gap (duality measure) given by

$$\mu = s_1 y + \boldsymbol{s}_2^T \boldsymbol{x}, \tag{36}$$

and $\mu^{pred}$ is the complementary gap that would be obtained if a full predictor step were taken, i.e.,

$$\mu^{pred} = (y + \alpha^{pred} \Delta y^{pred})(s_1 + \alpha^{pred} \Delta s_1^{pred}) + (\boldsymbol{x} + \alpha^{pred} \Delta \boldsymbol{x}^{pred})^T (\boldsymbol{s}_2 + \alpha^{pred} \Delta \boldsymbol{s}_2^{pred}) \tag{37}$$

Therefore, the centering component is computed by solving the following system:

$$\nabla_v \boldsymbol{F(v)} \boldsymbol{\Delta v}^{ce} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\sigma\mu \\ -\sigma\mu e \end{pmatrix}$$

*Corrector component*. It considers the curvature that the predictor component omits – *i.e.*, it is the second order approximation – to drive the complementary conditions (27) and (28) to zero so that the KKT's conditions is satisfied. This component is computed by solving the following system:

$$\nabla_v \boldsymbol{F(v)} \boldsymbol{\Delta v}^{co} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\Delta S_1 \Delta Y \\ -\Delta \boldsymbol{S}_2 \Delta \mathbf{X} \end{pmatrix}$$

Once the whole direction, $\Delta v$, for the current point has been computed, a maximum stepsize is determined with a security factor, $\eta$, to accelerate the convergence, i.e.,

$$\alpha^p = \eta min\Big\{1, \min_{y, x_i < 0} \Big( -\frac{y}{\Delta y}, -\frac{x_i}{\Delta x_i} \Big)\Big\} \tag{38}$$

$$\alpha^d = \eta min\Big\{1, \min_{s_1, s_{2i} < 0} \Big( -\frac{s_1}{\Delta s_1}, -\frac{s_{2i}}{\Delta s_{2i}} \Big)\Big\}, \tag{39}$$

The stepsizeto be used in the current iteration is given by

$$\alpha = min\{\alpha^p, \alpha^d\} \tag{40}$$

Finally, knowing the stepsize and the three component for the direction, the current point is updated as follows:

$$y = y + \alpha \Delta y \tag{41}$$

$$\boldsymbol{x} = \boldsymbol{x} + \alpha \Delta \boldsymbol{x} \tag{42}$$

$$\lambda_1 = \lambda_1 + \alpha \Delta \lambda_1 \tag{43}$$

$$\boldsymbol{\lambda}_2 = \boldsymbol{\lambda_2} + \alpha \Delta \boldsymbol{\lambda}_2 \tag{44}$$

Therefore, given a initial point $v^0$ which satisfies the nonnegativity constraints, the above – primal-dual – process is iteratively applied until the convergence criteria is satisfied. A criteria for convergence is to drive the complementary gap, $\mu$, within a permissible tolerance, say, 1e-10.