Statistical machine learning

where

Data are evidence (instantiations of random variables)

Hypotheses are probabilistic theories of how the domain works

Chapters 18 + 20

Overview: aims

- Understand the application of Bayes’ rule for learning from data and that all available hypotheses are considered
- Understand the Naïve Bayes classifier so well that you could implement it

Overview: topics

- Review: Uncertain knowledge, Bayes' rule
- Bayesian learning
- Maximum a posteriori (MAP) and Maximum likelihood (ML)
- Bayesian networks
- Naïve Bayes (including a worked example)

Uncertain knowledge: Review

- Prior probability
  - $P(\alpha)$ is the probability of a proposition α in the absence of any other information
- Joint probability distribution
  - A full joint distribution covers the complete set of random variables
- Conditional probability
  - $P(\alpha|\beta)$ is the probability of α given that all we know is β
- Independence
  - If variables are independent, knowledge of one does not influence knowledge of another

Bayes' rule: Review (1)

Bayes' rule: $$P(\beta \mid \alpha) = \frac{P(\alpha \mid \beta) P(\beta)}{P(\alpha)}$$

Multivalued variables: $$\mathrm{P}(Y \mid X) = \frac{\mathrm{P}(X \mid Y)\mathrm{P}(Y)}{\mathrm{P}(X)}$$

Given evidence, e: $$\mathrm{P}(Y \mid X, e) = \frac{\mathrm{P}(X \mid Y, e)\mathrm{P}(Y \mid e)}{\mathrm{P}(X \mid e)}$$

Bayes' rule: Review (2)

- Classification
- Diagnosis
- Finding one of the terms when you have the other three

$$P(\beta \mid \alpha) = \frac{P(\alpha \mid \beta) P(\beta)}{P(\alpha)}$$

Bayesian learning (1)

- *Data* are evidence (instantiations of the random variables that describe the domain)
- *Hypotheses* are probabilistic theories of how the domain works

Bayesian learning (2)

- Calculates the probability of each available hypothesis (model or input to output function), given the data
- Predicts output using *all* available hypotheses, weighted by their probabilities – not just using a *single* hypothesis (as in e.g. ID3 decision trees)
- Learning is thus reduced to probabilistic inference
- A limited set of hypotheses are considered

Bayesian learning (3)

- Assume we have (or choose) a finite number of hypotheses indexed by *i*
- Let *D* be a random variable representing the whole dataset and call the actual (observed) dataset ***d***
- Standard statistics notation: variable in upper case, observations of it in lower case
- Then the posterior probability of any hypothesis, h_i, is given by: (using Bayes' rule)

$$P(h_i \mid \mathbf{d}) = \frac{P(\mathbf{d} \mid h_i) P(h_i)}{P(\mathbf{d})}$$

Bayesian learning (4)

$$P(h_i \mid \mathbf{d}) = \frac{P(\mathbf{d} \mid h_i)P(h_i)}{P(\mathbf{d})} \propto P(\mathbf{d} \mid h_i)P(h_i)$$

$$P(\mathbf{d}) = \sum_i P(\mathbf{d} \mid h_i)P(h_i)$$

A normalising constant which is the same for all hypotheses – it ensures that all the probabilities add up to 1

Bayesian learning (5)

$$P(h_i \mid \mathbf{d}) \propto P(\mathbf{d} \mid h_i) P(h_i)$$

- $P(h_i|\mathbf{d})$: *likelihood* of the hypothesis h_i given the set of data $\mathbf{d}$
- $P(\mathbf{d}|h_i)$: *likelihood* of the data $\mathbf{d}$ given hypothesis h_i.
  - We should be able to calculate the likelihood for almost any hypothesis
- $P(h_i)$: *prior* probability of hypothesis h_i (without seeing any data)
  - We have to work this out from prior knowledge or experience
  - If we don't have any, we might make it the same for each hypothesis

Bayesian learning: Candy bags (1)

- Very large candy bags, either
  - h_1: 100% cherry
  - h_2: 75% cherry and 25% lime
  - h_3: 50% cherry and 50% lime
  - h_4: 25% cherry and 75% lime
  - h_5: 100% lime
- These hypotheses are discrete distributions
- Prior probabilities (here we know them)
  - $P(h_1)=0.10$, $P(h_2)=0.20$, $P(h_3)=0.40$, $P(h_4)=0.20$, $P(h_5)=0.10$
- Task: What kind of bag do we have ?

Bayesian learning: Candy bags (2)

- $P(h_1)=0.10$
- $P(h_2)=0.20$
- $P(h_3)=0.40$
- $P(h_4)=0.20$
- $P(h_5)=0.10$
  - If have no data, choose h_3 – it's the most likely
- Data: observing the candy in the bag by opening wrappers one by one
- Task: What kind of bag do we have?
  - Choose the correct hypothesis given some data

Bayesian learning: Candy bags (3)

$$P(\alpha \wedge \beta) = P(\alpha \mid \beta)P(\beta)$$

Assumption

$$P(\alpha \mid \beta) = P(\alpha)$$

$$P(\alpha \wedge \beta) = P(\alpha)P(\beta)$$

- Assume the observations (opening wrappers one by one) are *independently and identically distributed (i.i.d.)* ie: the probabilities for each candy observation don't depend on the previous ones and don't change
- So likelihood becomes:

$$P(\mathbf{d} \mid h_i) = \prod_j P(d_j \mid h_i)$$

Candy bags:
What kind of bag do we have? (1)

- $P(h_i|\boldsymbol{d}) \propto P(\boldsymbol{d}|h_i)\, P(h_i)$
- We have $P(h_i)$
- Calculate $P(\boldsymbol{d}|h_i)$ on the basis of observed data

h1: 100% cherry	P(h1)=0.10
h2: 75% cherry and 25% lime	P(h2)=0.20
h3: 50% cherry and 50% lime	P(h3)=0.40
h4: 25% cherry and 75% lime	P(h4)=0.20
h5: 100% lime	P(h5)=0.10

Candy bags: What kind of bag do we have? (2)

$$P(\mathbf{d} \mid h_1) = P(\text{lime} \mid h_1) = 0.00$$
$$P(\mathbf{d} \mid h_2) = P(\text{lime} \mid h_2) = 0.25$$
$$P(\mathbf{d} \mid h_3) = P(\text{lime} \mid h_3) = 0.50$$
$$P(\mathbf{d} \mid h_4) = P(\text{lime} \mid h_4) = 0.75$$
$$P(\mathbf{d} \mid h_5) = P(\text{lime} \mid h_5) = 1.00$$

$$P(h_i \mid \boldsymbol{d}) \propto \boxed{P(\boldsymbol{d} \mid h_i)} P(h_i)$$

Candy bags: What kind of bag do we have? (3)

$$P(\mathbf{d} \mid h_1) = P(\text{lime} \mid h_1)P(\text{lime} \mid h_1) = 0.00$$
$$P(\mathbf{d} \mid h_2) = P(\text{lime} \mid h_2)P(\text{lime} \mid h_2) = 0.06$$
$$P(\mathbf{d} \mid h_3) = P(\text{lime} \mid h_3)P(\text{lime} \mid h_3) = 0.25$$
$$P(\mathbf{d} \mid h_4) = P(\text{lime} \mid h_4)P(\text{lime} \mid h_4) = 0.56$$
$$P(\mathbf{d} \mid h_5) = P(\text{lime} \mid h_5)P(\text{lime} \mid h_5) = 1.00$$

$$P(h_i \mid \boldsymbol{d}) \propto P(\boldsymbol{d} \mid h_i)P(h_i)$$

Candy bags:
What kind of bag do we have? (4)

$$P(\mathbf{d} \mid h_1) = P(\text{lime} \mid h_1)P(\text{lime} \mid h_1)P(\text{lime} \mid h_1) = 0.00$$
$$P(\mathbf{d} \mid h_2) = P(\text{lime} \mid h_2)P(\text{lime} \mid h_2)P(\text{lime} \mid h_2) = 0.02$$
$$P(\mathbf{d} \mid h_3) = P(\text{lime} \mid h_3)P(\text{lime} \mid h_3)P(\text{lime} \mid h_3) = 0.13$$
$$P(\mathbf{d} \mid h_4) = P(\text{lime} \mid h_4)P(\text{lime} \mid h_4)P(\text{lime} \mid h_4) = 0.42$$
$$P(\mathbf{d} \mid h_5) = P(\text{lime} \mid h_5)P(\text{lime} \mid h_5)P(\text{lime} \mid h_5) = 1.00$$

$$P(h_i \mid \boldsymbol{d}) \propto \boxed{P(\boldsymbol{d} \mid h_i)} P(h_i)$$

Candy bags:
What kind of bag do we have? (5)

$$P(\mathbf{d} \mid h_1) = [P(\text{lime} \mid h_1)]^5 = 0.00$$

$$P(\mathbf{d} \mid h_2) = [P(\text{lime} \mid h_2)]^5 = 0.00$$

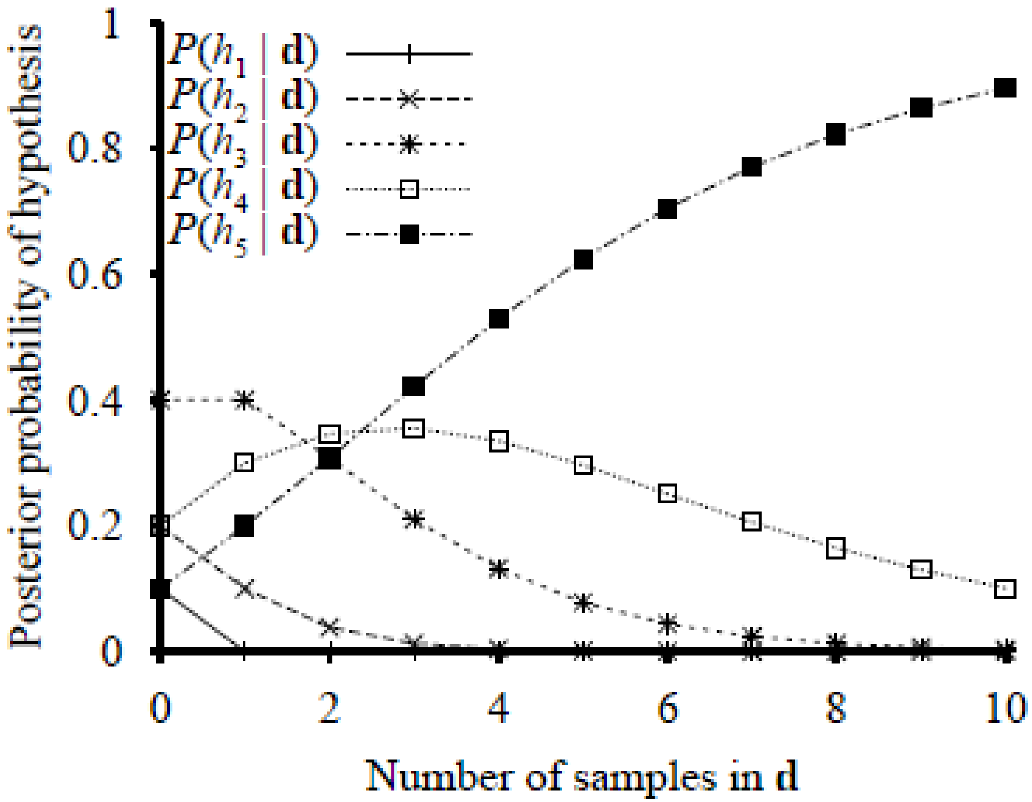
$$P(\mathbf{d} \mid h_3) = [P(\text{lime} \mid h_3)]^5 = 0.03$$

$$P(\mathbf{d} \mid h_4) = [P(\text{lime} \mid h_4)]^5 = 0.24$$

$$P(\mathbf{d} \mid h_5) = [P(\text{lime} \mid h_5)]^5 = 1.00$$

$$P(h_i \mid \boldsymbol{d}) \propto P(\boldsymbol{d} \mid h_i) P(h_i)$$

Candy bags:
What kind of bag do we have? (6)



- **d** = 0 to 10 lime candy observations

$$P(h_i \mid \boldsymbol{d}) = \alpha P(\boldsymbol{d} \mid h_i) P(h_i)$$

Bayesian learning: Candy bags (4)

- Task 2: predict the next observation d'
- Note that for Bayesian learning, all hypotheses are being used to predict

Candy bags: Predict d' (1)

$$P(d'|\mathbf{d}) = \sum_i P(d', h_i | \mathbf{d})$$

$$= \sum_i P(d'|h_i, \mathbf{d}) P(h_i | \mathbf{d})$$

$$= \sum_i P(d'|h_i) P(h_i | \mathbf{d})$$

$$\propto \sum_i P(d'|h_i) P(h_i) \prod_{j=1}^{n} P(d_j | h_i)$$

- Each hypothesis h_i determines a probability distribution over possible outcomes like d'
- The result is a weighted average over hypothesis-specific predictions

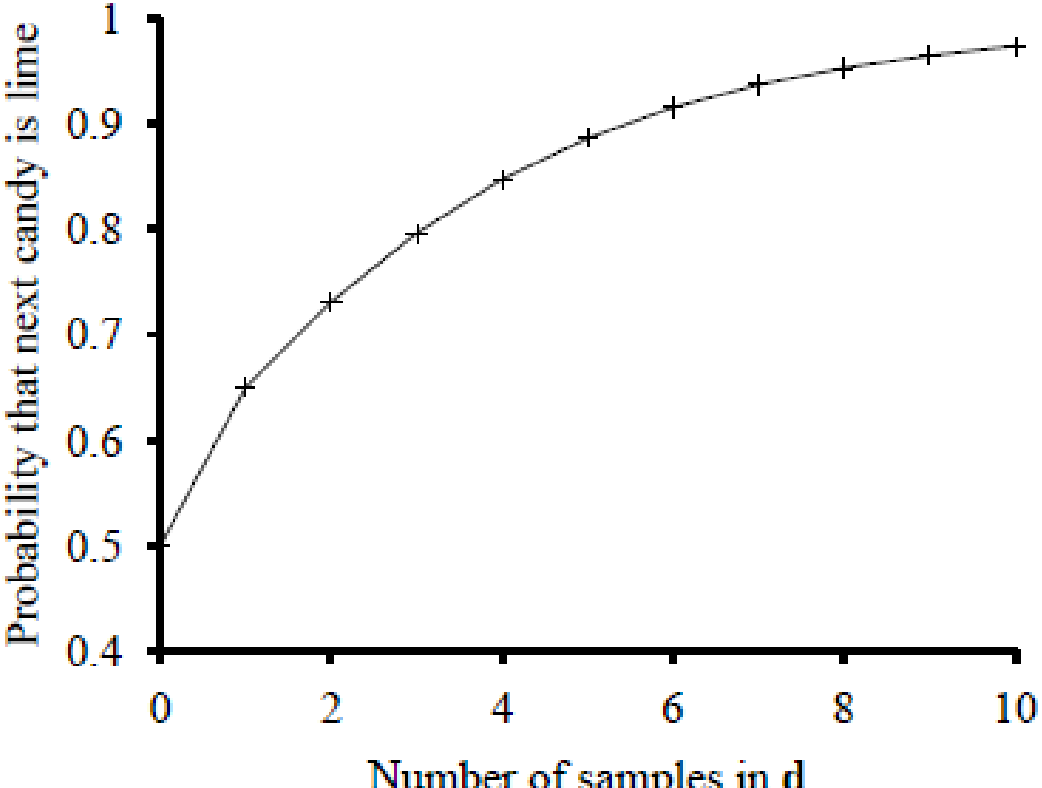
Candy bags: Predict d' (2)

$$P(d'|\mathbf{d}) \propto \sum_i P(d'|h_i)P(h_i)\prod_{j=1}^{n} P(d_j|h_i)$$

- We can calculate all these terms
- E.g. after ***d***=5 lime candies,
- P(*d'*=lime|***d***)$\propto$0(.1)0^5+.25(.2)$(.25)^5$+.5(.4)$(.5)^5$
  +.75(.2)$(.75)^5$+1(.1)1^5=0.142
- P(*d'*=cherry|***d***)$\propto$1(.1)0^5+.75(.2)$(.25)^5$+.5(.4)$(.5)^5$
  +.25(.2)$(.75)^5$+0(.1)1^5=0.018
- Can ignore normalising: lime is our prediction
- If normalising: P(*d'*=lime|***d***) = 0.142/(0.142+0.018) = 0.888
- Hence P(*d'*=cherry|***d***) = 1-0.888 = 0.112

Candy bags: Predict d' (3)

$$P(d'|\mathbf{d}) \propto \sum_i P(d'|h_i)P(h_i)\prod_{j=1}^{n} P(d_j|h_i)$$

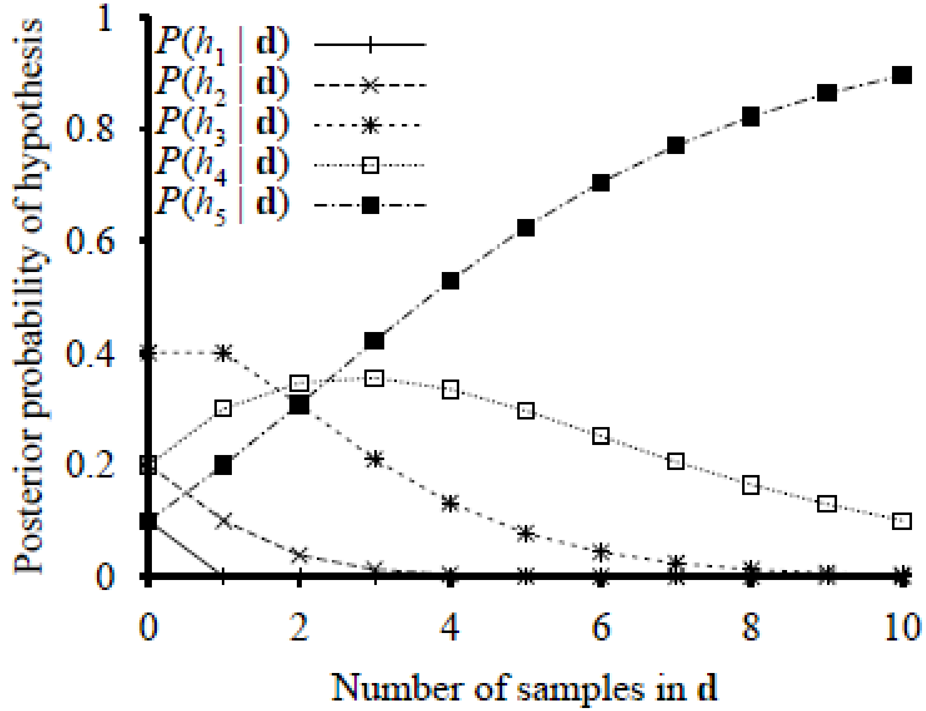


Bayesian learning

- If we have the full set of possible hypotheses and the correct prior probability for each, Bayesian prediction makes the optimal decision (regardless of the number of data points)
- We can get the predicted quantity and its estimated probability, given the data, and also get the (estimated) probabilities of each hypothesis
- But the space of hypotheses is usually very large – it can be useful to use approximations
  - Maximum a posteriori (MAP)
  - Maximum likelihood (ML)

Maximum a posteriori (MAP)

- Make predictions based on the most probable hypothesis only
  - Notes on example: h_{MAP} becomes h_5 after only three observations, but e.g. $P(d'|\boldsymbol{d}) \approx P(d'|h_5)$ takes many more observations



Maximum likelihood (ML)

- Assume a uniform prior on hypotheses, and ignore the prior
  - In general, h_{MAP} becomes h_{ML} as more data is collected and the impact of the prior fades

Parameter learning

- Predict a class variable based on attribute variables
- Naïve Bayes classifier
- (Bayesian networks)

Bayesian networks (1) (Chapter 14)

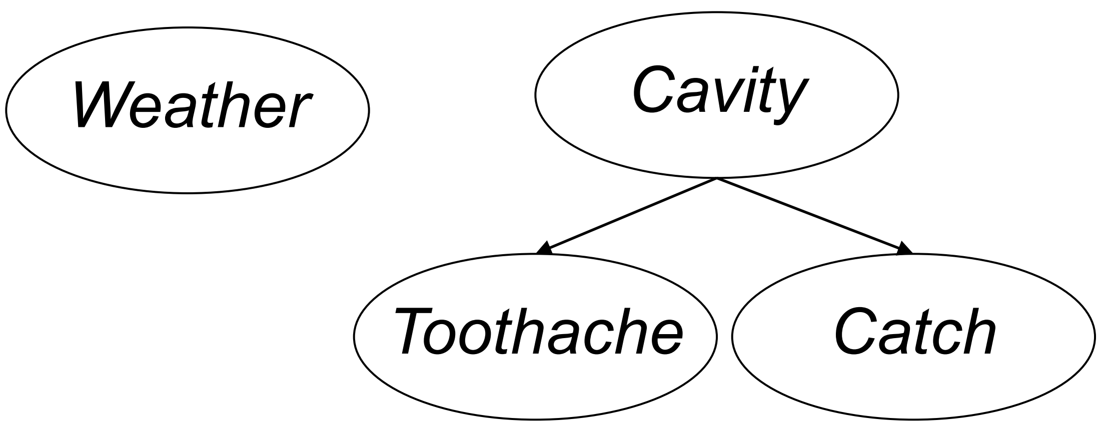
- Full joint probability distribution can answer any question about the domain
- Independence and conditional independence relationships between variables reduces the probabilities that need to be specified
- Bayesian networks are a way to represent the independence and conditional independence relationships between variables

Bayesian networks (2)

- Independence
  - If variables are independent, knowledge of one does not affect knowledge of another
- Conditional independence
  - Variables are conditionally independent if they are independent given the presence or absence of another variable

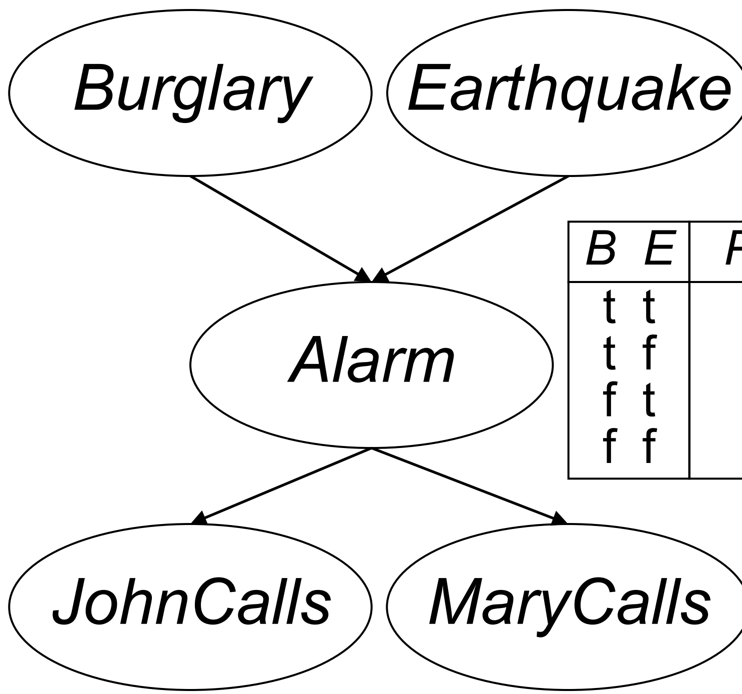
Bayesian networks (3)

- Directed acyclic graph where each node has quantitative probability information
- Nodes are the set of random variables, and are connected by links that specify influences



Bayesian networks (4)

- Conditional probabilities can be defined for each



P(Burglary)
0.001

P(Earthquake)
0.002

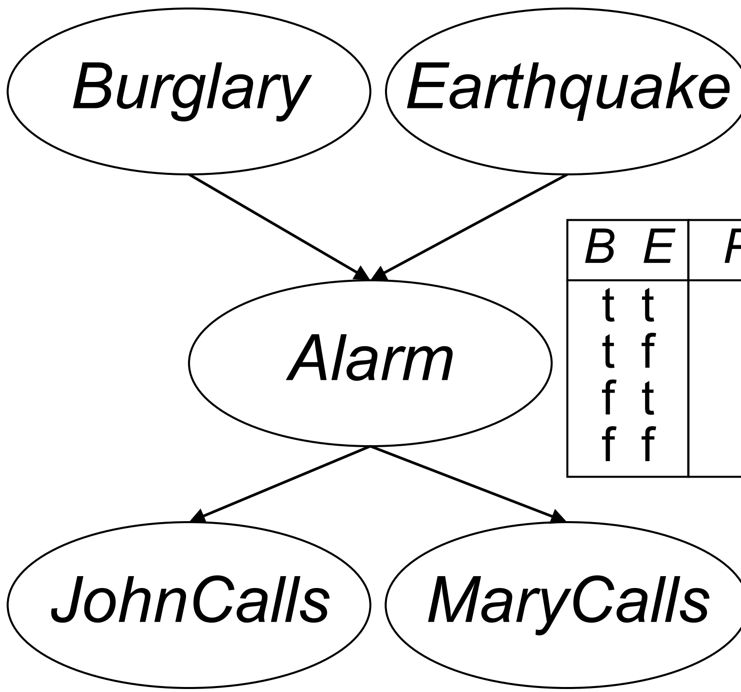
B	*E*	*P(Alarm)*
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	*P(JohnCalls)*
t	0.90
f	0.05

A	*P(MaryCalls)*
t	0.70
f	0.01

Bayesian networks (5)

- What is the probability that a burglary has occurred but not an Earthquake, the alarm has sounded, and both John and Mary have called?



P(Burglary)
0.001

P(Earthquake)
0.002

B	*E*	*P(Alarm)*
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

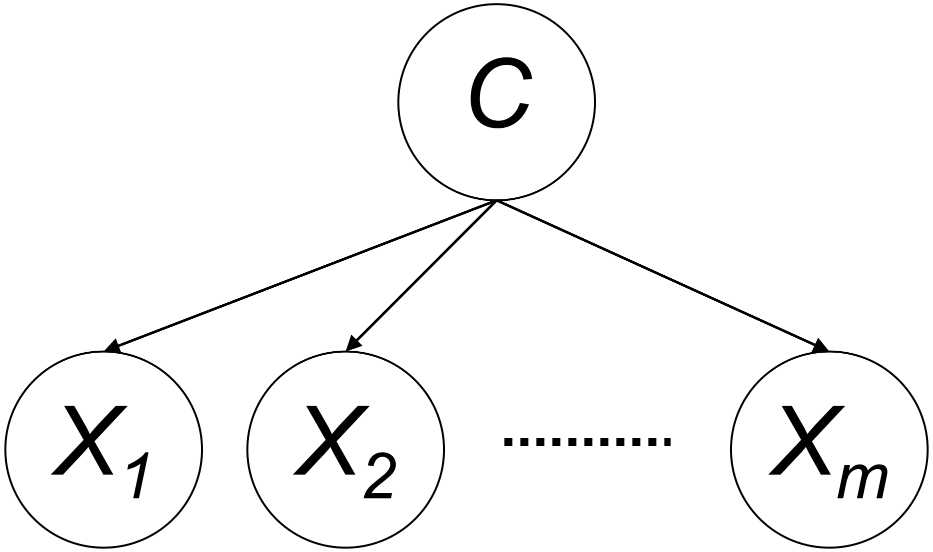
A	*P(JohnCalls)*
t	0.90
f	0.05

A	*P(MaryCalls)*
t	0.70
f	0.01

$$P(B \wedge \neg E \wedge A \wedge J \wedge M) = P(B)P(\neg E)P(A|B \wedge \neg E)P(J|A)P(M|A)$$

Bayesian networks: Naïve Bayes

- Bayesian network model in which the *Class* variable C (to be predicted) is the root and the *Attribute* variables X_i are the leaves
- Naïve: assumption that attributes are conditionally independent, given the class



Naïve Bayes model

- We wish to predict a class variable *C using* attribute variables X_i, $i=1,..,m$.
- Assume we know all the possible classes - usually only a small number e.g. 2-30.
- Assume we have some training data (index $j=1,..,n$) from which we can estimate, e.g. $P(X_i|C)$ and $P(C)$.
- The estimation of class probabilities is naïve because it assumes that all the attributes are conditionally independent of each other

$$P(C \mid X_1, X_2, \ldots, X_m) = \alpha P(C) \prod_{i=1}^{m} P(X_i \mid C)$$

Naïve Bayes: using training data (1)

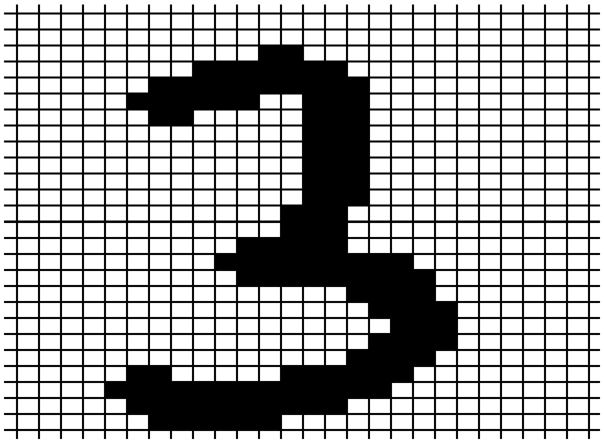
- Assume we have a training dataset $\boldsymbol{d}=\{x_j, c_j, j=1,..,n\}$ of n observations, where each $x_j = \{x_{ij}, i=1,..,m\}$ is a vector of m attribute values x_{ij}, and c_j is the correct label (class) for x_j (e.g. given by a human expert).
- If we get a new input vector $x_{new}= \{x_{i,new}\}$ and want to predict its class c_{new}, how is this done with Naïve Bayes?

Naïve Bayes: using training data (2)

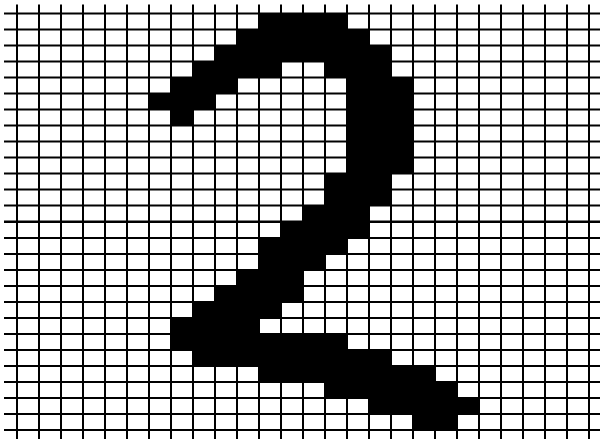
- Calculate the posterior probabilities for each possible class and choose the most likely. All quantities on the last row can be estimated from the training data. The |***d*** part is sometimes not mentioned.

$$P(c_{new} \mid x_{new}, \mathrm{d})$$
$$\propto P(x_{new} \mid c_{new}, \mathrm{d}) P(c_{new} \mid \mathrm{d})$$
$$\propto P(c_{new} \mid \mathrm{d}) \prod_{i=1}^{m} P(x_{i,new} \mid c_{new}, \mathrm{d})$$

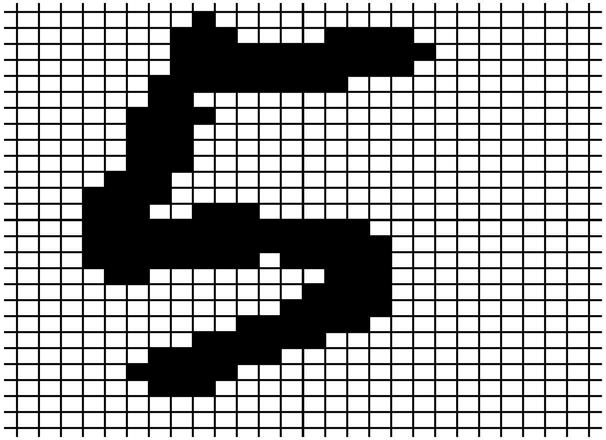
Naïve Bayes example: Number recognition (1)



C=3

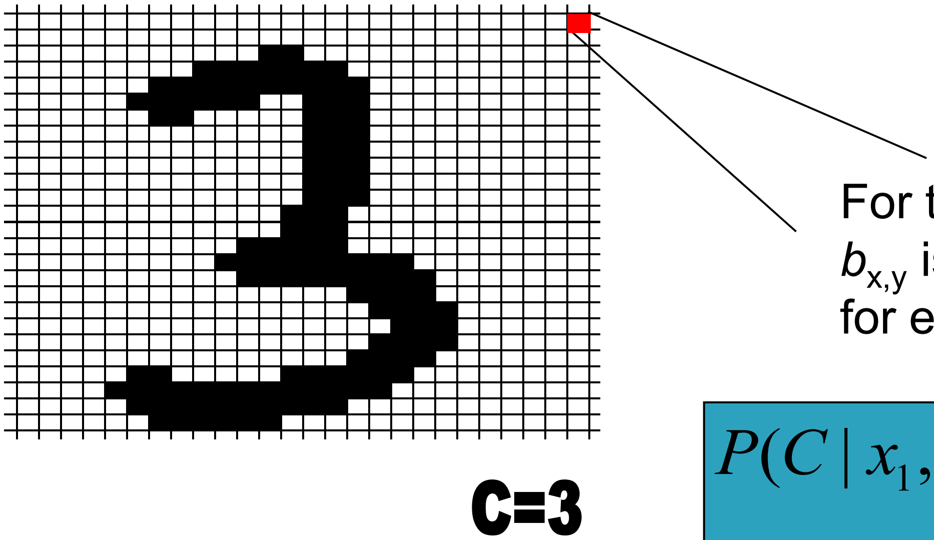


C=2



C=5

Naïve Bayes example: Number recognition (2)



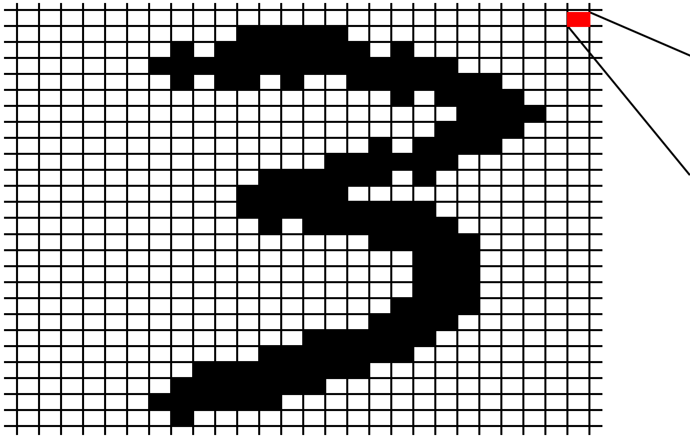
For this example *j*,
$b_{x,y}$ is on or off, i.e. add 0 or 1 to the count for each pixel per example.

$$P(C \mid x_1, x_2, \ldots, x_n) = \alpha P(C) \prod_i P(x_i \mid C)$$

- How to estimate $P(C)$ and $P(X_i|C)$ from training data ?

$$P(C=3) = count(C=3) / count(all)$$
$$P(x_i \mid C=3) = count(x_i \wedge C=3) / count(C=3)$$

Naïve Bayes example: Number recognition (3)



For this test example, $b_{x,y}$ is on or off, i.e. 0 or 1. Work out $P(b_{x,y}|C)$ for all x,y co-ordinates in image; can then predict C using Naive Bayes model.

C=?

- What if training sample counts are zero?
  - smooth distributions, i.e. redistribute some probability
- Conditional independence between input features?
- Problems with continuous input variables?
  - discretise or estimate using standard distributions, e.g. normal (see section 14.3 of textbook)

Naïve Bayes example: Restaurant problem (1)

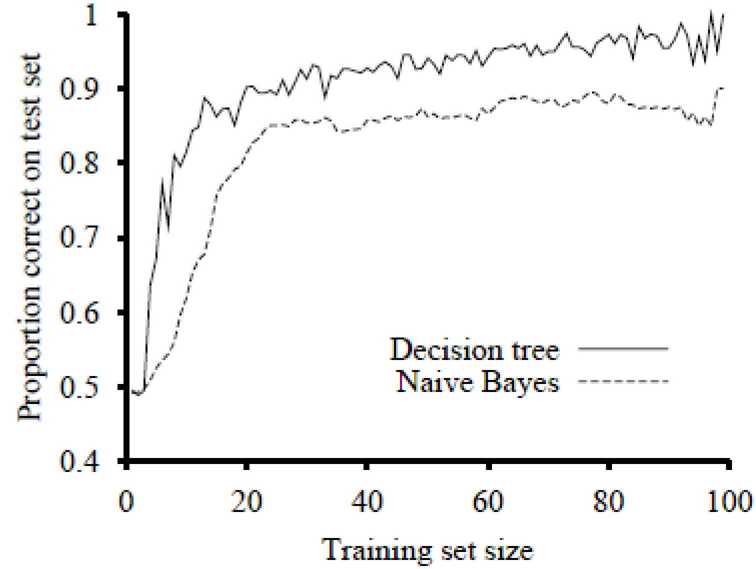


Figure 20.3 The learning curve for naive Bayes learning applied to the restaurant problem from Chapter 18; the learning curve for decision-tree learning is shown for comparison.

Naïve Bayes example: Restaurant problem (2)

	Fri/Sat	Hungry	Patrons	Type	Will wait?
1	FALSE	TRUE	Some	French	TRUE
2	FALSE	TRUE	Full	Thai	FALSE
3	FALSE	FALSE	Some	Burger	TRUE
4	TRUE	TRUE	Full	Thai	TRUE
5	TRUE	FALSE	Full	French	FALSE
6	FALSE	TRUE	Some	Italian	TRUE
7	FALSE	FALSE	None	Burger	FALSE
8	FALSE	TRUE	Some	Thai	TRUE
9	TRUE	FALSE	Full	Burger	FALSE
10	TRUE	TRUE	Full	Italian	FALSE
11	FALSE	FALSE	None	Thai	(FALSE)
12	TRUE	TRUE	Full	Burger	(TRUE)

Naïve Bayes example: Restaurant problem (3)

- Can construct the model and prediction as needed for each test pattern (11 and 12)
- Alternative is to build full Naïve Bayes model for every test pattern
- Can estimate probabilities from the training data (1-10)

$$P(c_{new} \mid x_{new}, \boldsymbol{d}) \propto P(c_{new} \mid \boldsymbol{d}) \prod_{i=1}^{m} P(x_{i,new} \mid c_{new}, \boldsymbol{d})$$

Naïve Bayes example: Restaurant problem (4)

$$P(c_{new} \mid x_{new}, \boldsymbol{d}) = \alpha P(c_{new} \mid \boldsymbol{d}) \prod_{i=1}^{m} P(x_{i,new} \mid c_{new}, \boldsymbol{d})$$

- c_{new}: possible class output for test example
- x_{new}: input variables for test example
- d: dataset of training examples

	Fri/Sat	Hungry	Patrons	Type	Will wait?
1	FALSE	TRUE	Some	French	TRUE
2	FALSE	TRUE	Full	Thai	FALSE
3	FALSE	FALSE	Some	Burger	TRUE
4	TRUE	TRUE	Full	Thai	TRUE
5	TRUE	FALSE	Full	French	FALSE
6	FALSE	TRUE	Some	Italian	TRUE
7	FALSE	FALSE	None	Burger	FALSE
8	FALSE	TRUE	Some	Thai	TRUE
9	TRUE	FALSE	Full	Burger	FALSE
10	TRUE	TRUE	Full	Italian	FALSE
11	FALSE	FALSE	None	Thai	(FALSE)
12	TRUE	TRUE	Full	Burger	(TRUE)

Naïve Bayes example: Restaurant problem (5)

$$P(c_{new} \mid x_{new}, \boldsymbol{d}) = \alpha P(c_{new} \mid \boldsymbol{d}) \prod_{i=1}^{m} P(x_{i,new} \mid c_{new}, \boldsymbol{d})$$

- $P(c_{new}|x_{new},\boldsymbol{d})$ values sum to 1 over the possible classes of c_{new}
- Renormalisation of $P(c_{new}|x_{new},\boldsymbol{d})$ values can be performed after un-normalised values are calculated

Naïve Bayes example: Restaurant problem (6)

$$P(c_{new} \mid x_{new}, \boldsymbol{d}) = \alpha P(c_{new} \mid \boldsymbol{d}) \prod_{i=1}^{m} P(x_{i,new} \mid c_{new}, \boldsymbol{d})$$

- P(WillWait=true) = 5/10 = 0.5
- P(WillWait=false) = 1-P(WillWait=true) = 0.5

	Fri/Sat	Hungry	Patrons	Type	Will wait?
1	FALSE	TRUE	Some	French	TRUE
2	FALSE	TRUE	Full	Thai	FALSE
3	FALSE	FALSE	Some	Burger	TRUE
4	TRUE	TRUE	Full	Thai	TRUE
5	TRUE	FALSE	Full	French	FALSE
6	FALSE	TRUE	Some	Italian	TRUE
7	FALSE	FALSE	None	Burger	FALSE
8	FALSE	TRUE	Some	Thai	TRUE
9	TRUE	FALSE	Full	Burger	FALSE
10	TRUE	TRUE	Full	Italian	FALSE

Naïve Bayes example: Restaurant problem (7)

$$P(c_{new} \mid x_{new}, \boldsymbol{d}) = \alpha P(c_{new} \mid \boldsymbol{d}) \prod_{i=1}^{m} P(x_{i,new} \mid c_{new}, \boldsymbol{d})$$

- P(Fri/Sat=true|WillWait=true) = 1/5 = 0.2
- P(Fri/Sat=false|WillWait=true) = 4/5 = 0.8
- P(Fri/Sat=true|WillWait=false) = 3/5 = 0.6
- P(Fri/Sat=false|WillWait=false) = 2/5 = 0.4

	Fri/Sat	Hungry	Patrons	Type	Will wait?
1	FALSE	TRUE	Some	French	TRUE
2	FALSE	TRUE	Full	Thai	FALSE
3	FALSE	FALSE	Some	Burger	TRUE
4	TRUE	TRUE	Full	Thai	TRUE
5	TRUE	FALSE	Full	French	FALSE
6	FALSE	TRUE	Some	Italian	TRUE
7	FALSE	FALSE	None	Burger	FALSE
8	FALSE	TRUE	Some	Thai	TRUE
9	TRUE	FALSE	Full	Burger	FALSE
10	TRUE	TRUE	Full	Italian	FALSE

Naïve Bayes example: Restaurant problem (8)

$$P(c_{new} \mid x_{new}, \boldsymbol{d}) = \alpha P(c_{new} \mid \boldsymbol{d}) \prod_{i=1}^{m} P(x_{i,new} \mid c_{new}, \boldsymbol{d})$$

- P(WillWait=true|TestExample11) = αP(WillWait=true)
  P(Fri/Sat=false|WillWait=true)
  P(Hungry=false|WillWait=true)
  P(Patrons=none|WillWait=true)
  P(Type=Thai|WillWait=true)
- = α(0.5)(4/5)(1/5)(0/5)(2/5)
- = α0.0

	Fri/Sat	Hungry	Patrons	Type	Will wait?
1	FALSE	TRUE	Some	French	TRUE
2	FALSE	TRUE	Full	Thai	FALSE
3	FALSE	FALSE	Some	Burger	TRUE
4	TRUE	TRUE	Full	Thai	TRUE
5	TRUE	FALSE	Full	French	FALSE
6	FALSE	TRUE	Some	Italian	TRUE
7	FALSE	FALSE	None	Burger	FALSE
8	FALSE	TRUE	Some	Thai	TRUE
9	TRUE	FALSE	Full	Burger	FALSE
10	TRUE	TRUE	Full	Italian	FALSE
11	FALSE	FALSE	None	Thai	(FALSE)
12	TRUE	TRUE	Full	Burger	(TRUE)

Naïve Bayes example: Restaurant problem (9)

$$P(c_{new} \mid x_{new}, \boldsymbol{d}) = \alpha P(c_{new} \mid \boldsymbol{d}) \prod_{i=1}^{m} P(x_{i,new} \mid c_{new}, \boldsymbol{d})$$

- P(WillWait=false|TestExample11) = αP(WillWait=false)
  P(Fri/Sat=false|WillWait=false)
  P(Hungry=false|WillWait=false)
  P(Patrons=none|WillWait=false)
  P(Type=Thai|WillWait=false)
- = α(0.5)(2/5)(3/5)(1/5)(1/5)
- = α0.0048

	Fri/Sat	Hungry	Patrons	Type	Will wait?
1	FALSE	TRUE	Some	French	TRUE
2	FALSE	TRUE	Full	Thai	FALSE
3	FALSE	FALSE	Some	Burger	TRUE
4	TRUE	TRUE	Full	Thai	TRUE
5	TRUE	FALSE	Full	French	FALSE
6	FALSE	TRUE	Some	Italian	TRUE
7	FALSE	FALSE	None	Burger	FALSE
8	FALSE	TRUE	Some	Thai	TRUE
9	TRUE	FALSE	Full	Burger	FALSE
10	TRUE	TRUE	Full	Italian	FALSE
11	FALSE	FALSE	None	Thai	(FALSE)
12	TRUE	TRUE	Full	Burger	(TRUE)

Naïve Bayes example: Restaurant problem (10)

$$P(c_{new} \mid x_{new}, \boldsymbol{d}) = \alpha P(c_{new} \mid \boldsymbol{d}) \prod_{i=1}^{m} P(x_{i,new} \mid c_{new}, \boldsymbol{d})$$

- P(WillWait=true|TestExample11) = α0.0
- P(WillWait=false|TestExample11) = α0.0048
- Predicts false

	Fri/Sat	Hungry	Patrons	Type	Will wait?
1	FALSE	TRUE	Some	French	TRUE
2	FALSE	TRUE	Full	Thai	FALSE
3	FALSE	FALSE	Some	Burger	TRUE
4	TRUE	TRUE	Full	Thai	TRUE
5	TRUE	FALSE	Full	French	FALSE
6	FALSE	TRUE	Some	Italian	TRUE
7	FALSE	FALSE	None	Burger	FALSE
8	FALSE	TRUE	Some	Thai	TRUE
9	TRUE	FALSE	Full	Burger	FALSE
10	TRUE	TRUE	Full	Italian	FALSE
11	FALSE	FALSE	None	Thai	(FALSE)
12	TRUE	TRUE	Full	Burger	(TRUE)

Naïve Bayes example: Restaurant problem (11)

$$P(c_{new} \mid x_{new}, \boldsymbol{d}) = \alpha P(c_{new} \mid \boldsymbol{d}) \prod_{i=1}^{m} P(x_{i,new} \mid c_{new}, \boldsymbol{d})$$

- P(WillWait=true|TestExample12) = αP(WillWait=true) P(Fri/Sat=true|WillWait=true) P(Hungry=true|WillWait=true) P(Patrons=full|WillWait=true) P(Type=Burger|WillWait=true)
- = α(0.5)(1/5)(4/5)(1/5)(1/5)
- = α0.0032

	Fri/Sat	Hungry	Patrons	Type	Will wait?
1	FALSE	TRUE	Some	French	TRUE
2	FALSE	TRUE	Full	Thai	FALSE
3	FALSE	FALSE	Some	Burger	TRUE
4	TRUE	TRUE	Full	Thai	TRUE
5	TRUE	FALSE	Full	French	FALSE
6	FALSE	TRUE	Some	Italian	TRUE
7	FALSE	FALSE	None	Burger	FALSE
8	FALSE	TRUE	Some	Thai	TRUE
9	TRUE	FALSE	Full	Burger	FALSE
10	TRUE	TRUE	Full	Italian	FALSE
11	FALSE	FALSE	None	Thai	(FALSE)
12	TRUE	TRUE	Full	Burger	(TRUE)

Naïve Bayes example: Restaurant problem (12)

$$P(c_{new} \mid x_{new}, \boldsymbol{d}) = \alpha P(c_{new} \mid \boldsymbol{d}) \prod_{i=1}^{m} P(x_{i,new} \mid c_{new}, \boldsymbol{d})$$

- P(WillWait=false|TestExample12) = αP(WillWait=false) P(Fri/Sat=true|WillWait=false) P(Hungry=true|WillWait=false) P(Patrons=full|WillWait=false) P(Type=Burger|WillWait=false)
- = α(0.5)(3/5)(2/5)(4/5)(2/5)
- = α0.0384

	Fri/Sat	Hungry	Patrons	Type	Will wait?
1	FALSE	TRUE	Some	French	TRUE
2	FALSE	TRUE	Full	Thai	FALSE
3	FALSE	FALSE	Some	Burger	TRUE
4	TRUE	TRUE	Full	Thai	TRUE
5	TRUE	FALSE	Full	French	FALSE
6	FALSE	TRUE	Some	Italian	TRUE
7	FALSE	FALSE	None	Burger	FALSE
8	FALSE	TRUE	Some	Thai	TRUE
9	TRUE	FALSE	Full	Burger	FALSE
10	TRUE	TRUE	Full	Italian	FALSE
11	FALSE	FALSE	None	Thai	(FALSE)
12	TRUE	TRUE	Full	Burger	(TRUE)

Naïve Bayes example: Restaurant problem (13)

$$P(c_{new} \mid x_{new}, \boldsymbol{d}) = \alpha P(c_{new} \mid \boldsymbol{d}) \prod_{i=1}^{m} P(x_{i,new} \mid c_{new}, \boldsymbol{d})$$

- P(WillWait=true|TestExample12) = α0.0032
- P(WillWait=false|TestExample12) = α0.0384
- Predicts false

	Fri/Sat	Hungry	Patrons	Type	Will wait?
1	FALSE	TRUE	Some	French	TRUE
2	FALSE	TRUE	Full	Thai	FALSE
3	FALSE	FALSE	Some	Burger	TRUE
4	TRUE	TRUE	Full	Thai	TRUE
5	TRUE	FALSE	Full	French	FALSE
6	FALSE	TRUE	Some	Italian	TRUE
7	FALSE	FALSE	None	Burger	FALSE
8	FALSE	TRUE	Some	Thai	TRUE
9	TRUE	FALSE	Full	Burger	FALSE
10	TRUE	TRUE	Full	Italian	FALSE
11	FALSE	FALSE	None	Thai	(FALSE)
12	TRUE	TRUE	Full	Burger	(TRUE)

Summary

- Review of uncertain knowledge and Bayes rule
- Bayesian Learning
- Maximum a posteriori (MAP) learning
- Maximum likelihood (ML) learning
- Bayesian networks
- Naïve Bayes