COMS3100/7100

Introduction to Communications

Lecture 4: LTI Systems

This lecture:

1. The Impulse Response & Convolution
2. Interconnections of LTI Systems
3. System Properties & the Impulse Response
4. The Step Response

Ref: HvV pp. 97–141, CCR ch. 3, Couch ch. 2

The Impulse Response & LTI

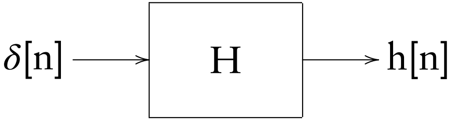
- One of the most important properties of an LTI system is that the system is characterised by its impulse response.
- Given the response of the system to an impulse, the response to any other signal can be computed in a straightforward manner.

The Impulse Response & not-LTI

- As the name suggests the impulse response is the response of a system given an impulse.
- All systems have this — but only in LTI systems does this allow us to characterise the response to other input signals using this.

The Discrete-Time Case

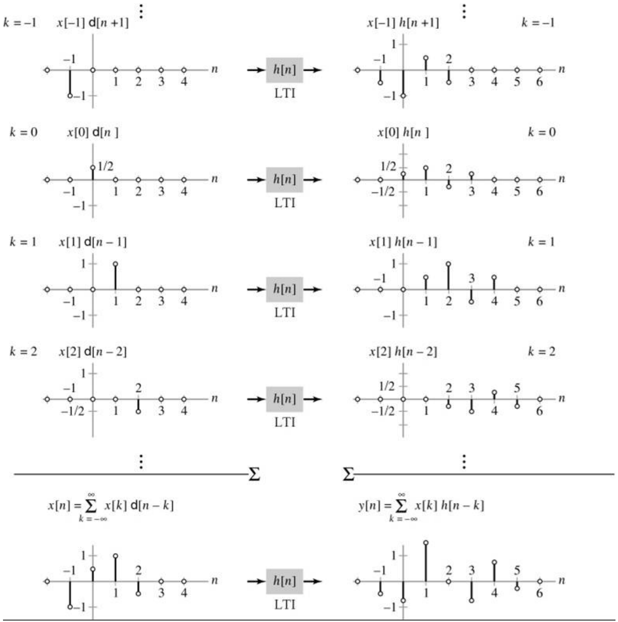
- Given an input impulse, let the output of the system — the discrete-time impulse response — be denoted $h[n]$.



Discrete-Time Case - Convolution

- The output of a discrete-time LTI system is the discrete-time linear convolution of the input with the impulse response. We write

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \quad (1)$$



k = −1
x[−1] d[n + 1]
x[−1] h[n + 1]
k = 0
x[0] d[n]
x[0] h[n]
k = 1
x[1] d[n − 1]
x[1] h[n − 1]
k = 2
x[2] d[n − 2]
x[2] h[n − 2]
h[n]
LTI
Σ
x[n] = Σ_{k=−∞}^{∞} x[k] d[n − k]
y[n] = Σ_{k=−∞}^{∞} x[k] h[n − k]
n

Convolution Example

- Find $y[n] = h[n] * x[n]$ where, $h[n] = \delta(t) + 3\delta(t+2)$ and $x[n] = n^2$.
- $y[0] = x[0]$
- $y[1] = x[1]$
- $y[2] = x[2] + 3x[0]$
- $y[3] = x[3] + 3x[1]$

Convolution Example

- Find $y[n] = h[n] * x[n]$ where, $h[n] = \delta(t) + 3\delta(t+2)$ and $x[n] = n^2$.
- $y[0] = x[0]$
- $y[1] = x[1]$
- $y[2] = x[2] + 3x[0]$
- $y[3] = x[3] + 3x[1]$

Convolution Example

- Find $y[n] = h[n] * x[n]$ where, $h[n] = \delta(t) + 3\delta(t+2)$ and $x[n] = n^2$.
- $y[0] = x[0]$
- $y[1] = x[1]$
- $y[2] = x[2] + 3x[0]$
- $y[3] = x[3] + 3x[1]$

Convolution Example

- Find $y[n] = h[n] * x[n]$ where, $h[n] = \delta(t) + 3\delta(t+2)$ and $x[n] = n^2$.
- $y[0] = x[0]$
- $y[1] = x[1]$
- $y[2] = x[2] + 3x[0]$
- $y[3] = x[3] + 3x[1]$

Convolution Example

- Find $y[n] = h[n] * x[n]$ where, $h[n] = \delta(t) + 3\delta(t+2)$ and $x[n] = n^2$.
- $y[0] = x[0]$
- $y[1] = x[1]$
- $y[2] = x[2] + 3x[0]$
- $y[3] = x[3] + 3x[1]$

The Continuous-Time Case

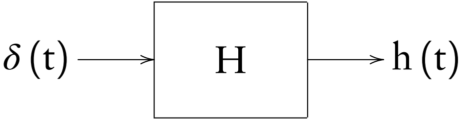
We can now develop the idea of an impulse response for a continuous-time LTI system in a nearly analogous fashion to discrete time.

- The following identity is the continuous-time analogue of (1):

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau.$$

Continuous-Time Case Properties

- If the input to a continuous-time LTI system is $\delta(t)$, label the output $h(t)$ — the continuous-time impulse response.



Continous-Time Case - Convolution

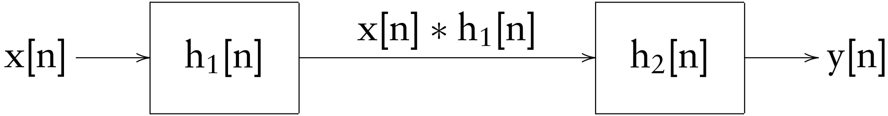
- The output is the continuous-time linear convolution of $x(t)$ and $h(t)$. We write

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) \, d\tau.$$

Interconnected LTI Systems

Cascaded Connection of LTI Systems

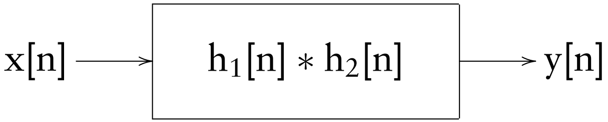
Suppose we cascade LTI systems with impulse responses $h_1[n]$, $h_2[n]$.



LTI Systems - Associativity

- We can effectively combine the two systems because convolution is associative, i.e.,

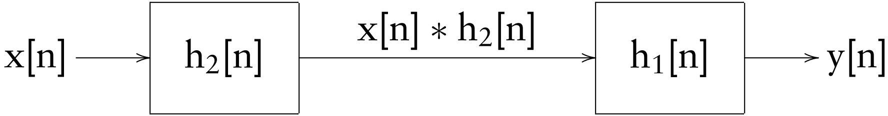
$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n]).$$



LTI Systems - Commutative

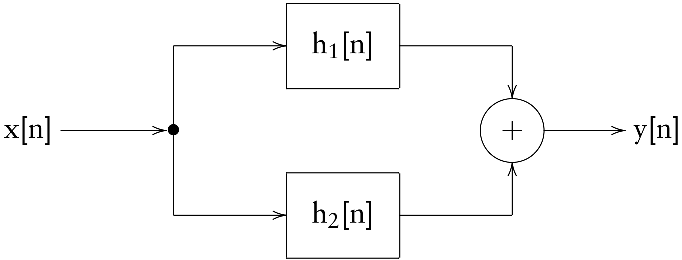
- We can also swap the order of cascaded LTI systems because convolution is commutative, i.e.,

$$h_1[n] * h_2[n] = h_2[n] * h_1[n].$$



Parallel Connection of LTI Systems

Suppose we connect two LTI systems in parallel.

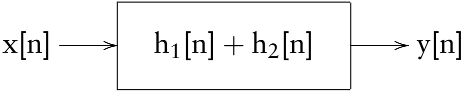


- Convolution is distributive over addition, i.e.,

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

LTI Systems - Combing

- Therefore, we can combine the two systems:



- Each of these results can be carried over into continuous time.

System Properties - Memory

Three of the properties previously examined for systems — namely memory, causality & stability — are reflected in the form of the impulse response in LTI systems.

- If an LTI system is memoryless then the impulse response must satisfy

$$h[n] = 0 \quad \text{when } n \neq 0 \qquad \text{or}$$

$$h(t) = 0 \quad \text{when } t \neq 0.$$

System Properties - Casual

- If an LTI system is causal then

$$h[n] = 0 \quad \text{when } n < 0 \qquad \text{or}$$

$$h(t) = 0 \quad \text{when } t < 0.$$

  - Some authors, e.g., CCR, also call any signal with the above properties causal. I prefer the term right-sided for these signals.

System Properties - Stability + Impulse Response

- If an LTI system is stable then the impulse response must be absolutely summable in the discrete-time case or absolutely integrable in the continuous-time case, i.e.,

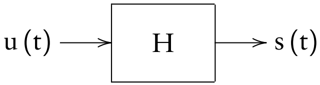
$$\sum_{n=\infty}^{\infty} |h[n]| < \infty \qquad \text{or} \qquad \int_{-\infty}^{\infty} |h(t)|\, dt < \infty.$$

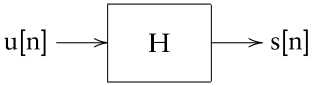
Step Response

Since impulses are difficult to generate and work with, real LTI systems (like RLC networks) are often experimentally characterised by measuring their step response instead.

Step Response (2)

- The step response is simply a system's output given a step input.





Step Response (2)

- For the discrete-time step response, we have

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k].$$

- Similarly, in continuous time,

$$s(t) = u(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau.$$

- Hence, the step response yields the impulse response since

$$h[n] = s[n] - s[n-1] \qquad \text{and} \qquad h(t) = \frac{d}{dt}s(t).$$