COMS3100/7100

Introduction to Communications

Lecture 5: Fourier Representations

This lecture:

1. Eigenfunctions of LTI Systems
2. Four Fourier Representations
3. Existence of Fourier Representations
4. Properties of Fourier Representations

Ref: HvV pp. 195–312, CCR ch. 2, Couch ch. 2.

Eigenfunctions of LTI Systems

In satisfying the equation

$$\mathrm{H}\{\mathrm{x}\,(\mathrm{t})\} = \lambda \mathrm{x}\,(\mathrm{t}),$$

for some system with operator H, the function $\mathrm{x}\,(\mathrm{t})$ is termed an eigenfunction and λ an eigenvalue of the system.

Eigenfunctions (2)

- The eigenfunctions are functions which are unchanged by the system from input to output, except in scale.
- Complex exponentials are eigenfunctions of LTI systems.

Eigenfunctions(3)

- To demonstrate, observe that the output to $x[n] = e^{j\omega n}$ is

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} = \underbrace{e^{j\omega n}}_{x[n]} \underbrace{\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}}_{\lambda} . \end{aligned}$$

- The eigenvalue λ is the frequency response of the system and we will label it $H(e^{j\omega})$.

Complex Exponential

Using Eigenfunctions we can use the following mapping.

$e^{j\omega t} = \cos \omega t + j \sin \omega t$

This gives the following Fourier Representation

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 n t}$$

Calculation of Fourier Series

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + jb_n \sin n\omega t)$$

where $\omega_0 = \frac{2\pi}{T}(\text{Fundamental Freq.}), c_n = a_n + jb_n$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t)\ dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega_0 t\ dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega_0 t\ dt$$

DC component, even and odd terms.

Simple Examples

Using Euler’s Relation we get:

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

Fourier Series Example: cosine

$$f(t) = \cos(\omega_0 t)$$

Converting to Euler Representation

$$f(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

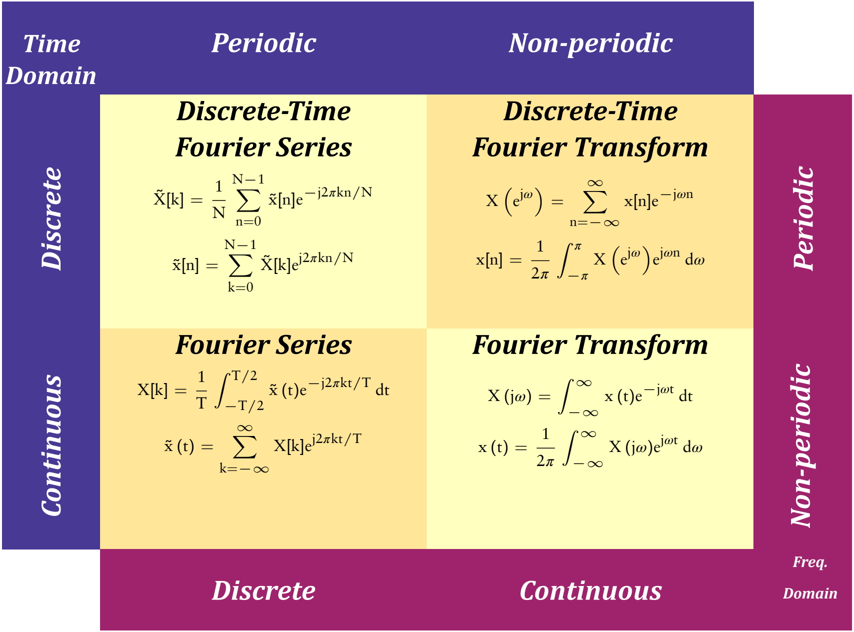
Hence,

$$c_1 = \tfrac{1}{2}, c_{-1} = \tfrac{1}{2}, \text{other } c_n = 0$$

Fourier Series Example: sine

$$f(t) = \sin(\omega_0 t)$$

Four Fourier Representations



Time Domain	*Periodic*	*Non-periodic*	
Discrete	***Discrete-Time Fourier Series*** $\tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j2\pi kn/N}$ $\tilde{x}[n] = \sum_{k=0}^{N-1}\tilde{X}[k]e^{j2\pi kn/N}$	***Discrete-Time Fourier Transform*** $X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}\,d\omega$	*Periodic*
Continuous	***Fourier Series*** $X[k] = \frac{1}{T}\int_{-T/2}^{T/2}\tilde{x}(t)e^{-j2\pi kt/T}\,dt$ $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j2\pi kt/T}$	***Fourier Transform*** $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$ $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$	*Non-periodic*
	Discrete	*Continuous*	*Freq. Domain*

Existence of the Fourier Representations

Under what conditions do the Fourier representations exist?

- The DTFS and its inverse always exist for any discrete-time periodic signal (after all, they are just finite sums).
- However, the other Fourier representations involve infinite sums or integrals, so we need to be careful with convergence.

Existence (2)

- As an example, consider the convergence properties of the DTFT.
- The DTFT converges uniformly if the signal is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

Existence (3)

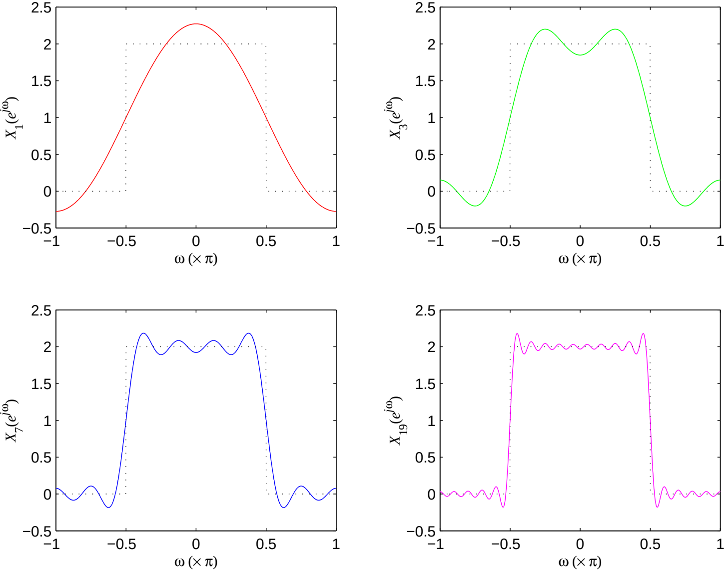
- On the other hand, the DTFT only converges in mean square if the signal is only square summable:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty.$$

- This leads to Gibbs' effect, now demonstrated for a sinc pulse.

Gibbs' Effect

gibbs.m, gibbs2.m



Fourier Properties - Linearity

Linearity. All Fourier representations exhibit a linearity, or superposition, property.

- For example, in the DTFS,

$$a\tilde{x}[n] + b\tilde{y}[n] \xleftrightarrow{\text{DTFS; } \omega_0} a\tilde{X}[k] + b\tilde{Y}[k].$$

Fourier Properties - Symmetry

Symmetry. If a signal is either entirely real or entirely imaginary then it obeys a symmetry property in the frequency domain.

- For example, if a periodic, continuous-time signal is real then its FS coefficients are conjugate symmetric.

Fourier Properties - Convolution

Convolution. Generally speaking, if we convolve two signals in the time domain then we multiply them in the frequency domain.

- For the FT and the DTFT, the convolution property refers to linear convolution, e.g.,

$$x[n] * y[n] \xleftrightarrow{\text{DTFT}} X\left(e^{j\omega}\right) Y\left(e^{j\omega}\right).$$

- For the FS and the DTFS, the convolution property refers to periodic convolution, e.g.,

$$\tilde{x}(t) \circledast \tilde{y}(t) \xleftrightarrow{\text{FS};\ \omega_0} T X[k] Y[k].$$

Fourier Properties - Convolution (2)

- Periodic convolution is like linear convolution, but calculated only over one period, i.e.,

$$\tilde{x}[n] \circledast \tilde{y}[n] = \sum_{k=0}^{N-1} \tilde{x}[k]\tilde{y}[n-k],$$

$$\tilde{x}(t) \circledast \tilde{y}(t) = \int_0^T \tilde{x}(\tau)\tilde{y}(t-\tau)\,d\tau.$$

Fourier Properties - Modulation

Modulation. Conversely, multiplication (modulation) in the time domain yields convolution in the frequency domain.

- For the FS and the DTFT the convolution is periodic.

Fourier Properties - Integration

Integration and Differentiation. Where the representation is continuous in one domain, integration and differentiation properties are admitted.

- For example, in the FT,

$$\frac{dx(t)}{dt} \overset{FT}{\longleftrightarrow} j\omega X(j\omega).$$

Fourier Properties - Duality

Duality. Where the time-domain and frequency-domain representations are either both continuous or both discrete, i.e., the DTFS and the FT, we observe a duality property.

- For example, in the DTFS,

$$\tilde{X}[n] \xleftrightarrow{\text{DTFS};\ \omega_0} \frac{1}{N}\tilde{x}[-k].$$

Fourier Properties - Time & Freq Shift

Time and Frequency Shift. If we shift a signal in time then we multiple its Fourier representation by a complex exponential, and vice versa.

- For example, in the FS,

$$\tilde{x}\left(t - t_0\right) \xleftrightarrow{\text{FS};\ \omega_0} e^{-jk\omega_0 t_0} X[k].$$

Fourier Properties - Parseval Rel.

Parseval Relationships. It is possible to calculate the energy or power of a signal in the frequency domain as well as in the time domain.

- For example, in the DTFT,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega.$$

Notes on CCR Presentation

Only two of the Fourier representations are discussed in any detail:

- Fourier transform
  - CCR use the notation $\mathrm{X}(\mathrm{f})$ or $\mathcal{F}[\mathrm{x}(\mathrm{t})]$ for the Fourier transform of $\mathrm{x}(\mathrm{t})$, rather than $\mathrm{X}(\mathrm{j}\omega)$, where $\omega = 2\pi\mathrm{f}$.
  - This affects the inverse Fourier transform, which is then:

$$\mathrm{x}(\mathrm{t}) = \int_{-\infty}^{\infty} \mathrm{X}(\mathrm{f}) e^{\mathrm{j}2\pi\mathrm{f}\mathrm{t}}\,\mathrm{d}\mathrm{f}$$

Notes on CCR Presentation

- Fourier series
  - The Fourier series presented here is what CCR calls the ‘exponential Fourier series’.
- Discrete-Time Fourier Series
  - The DTFS is mentioned only in passing on p. 44 in reference to the fast Fourier transform (FFT).
  - Like many authors, CCR instead call this the ‘Discrete Fourier Transform’ (DFT).

Tables of Fourier Properties

Discrete-Time Fourier Series

Property	***Time domain***	***Frequency domain***
Linearity	$a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
Duality	$\tilde{X}[n]$	$\frac{1}{N}\tilde{x}[-k]$
Time-shift	$\tilde{x}[n - n_0]$	$e^{-j2\pi k n_0/N}\tilde{X}[k]$
Frequency-shift	$e^{j2\pi k_0 n/N}\tilde{x}[n]$	$\tilde{X}[k - k_0]$
Convolution	$\tilde{x}_1[n] \circledast \tilde{x}_2[n]$	$N\tilde{X}_1[k]\tilde{X}_2[k]$
Modulation	$\tilde{x}_1[n]\tilde{x}_2[n]$	$\tilde{X}_1[k] \circledast \tilde{X}_2[k]$
Time-reversal	$\tilde{x}[-n]$	$\tilde{X}[-k]$
Conjugation	$\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
Symmetry (real)	$\mathfrak{Im}\{\tilde{x}[n]\} = 0$	$\tilde{X}[k] = \tilde{X}^*[-k]$
Symmetry (imag)	$\mathfrak{Re}\{\tilde{x}[n]\} = 0$	$\tilde{X}[k] = -\tilde{X}^*[-k]$
Parseval	$\sum_{n=0}^{N-1} \lvert\tilde{x}[n]\rvert^2 = N \sum_{k=0}^{N-1} \left\lvert\tilde{X}[k]\right\rvert^2$	

Tables of Fourier Properties

Discrete-Time Fourier Transform

Property	***Time domain***	***Frequency domain***
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
Differentiation (frequency)	$nx[n]$	$j\,\frac{dX(e^{j\omega})}{d\omega}$
Time-shift	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency-shift	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Convolution	$x_1[n] * x_2[n]$	$X_1(e^{j\omega})X_2(e^{j\omega})$
Modulation	$x_1[n]x_2[n]$	$\frac{1}{2\pi}X_1(e^{j\omega}) \circledast X_2(e^{j\omega})$
Time-reversal	$x[-n]$	$X(e^{-j\omega})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Symmetry (real)	$\mathfrak{Im}\{x[n]\} = 0$	$X(e^{j\omega}) = X^*(e^{-j\omega})$
Symmetry (imag)	$\mathfrak{Re}\{x[n]\} = 0$	$X(e^{j\omega}) = -X^*(e^{-j\omega})$
Parseval	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\lvert X(e^{j\omega})\right\rvert^2 \, d\omega$	

Properties of the Fourier Series

Property	***Time domain***	***Frequency domain***
Linearity	$a\tilde{x}_1(t) + b\tilde{x}_2(t)$	$aX_1[k] + bX_2[k]$
Differentiation (time)	$\frac{d\tilde{x}(t)}{dt}$	$\frac{j2\pi k}{T}X[k]$
Time-shift	$\tilde{x}(t - t_0)$	$e^{-j2\pi k t_0/T}X[k]$
Frequency-shift	$e^{j2\pi k_0 t/T}\tilde{x}(t)$	$X[k - k_0]$
Convolution	$\tilde{x}_1(t) \circledast \tilde{x}_2(t)$	$TX_1[k]X_2[k]$
Modulation	$\tilde{x}_1(t)\tilde{x}_2(t)$	$X_1[k] \circledast X_2[k]$
Time-reversal	$\tilde{x}(-t)$	$X[-k]$
Conjugation	$\tilde{x}^*(t)$	$X^*[-k]$
Symmetry (real)	$\mathfrak{Im}\{\tilde{x}(t)\} = 0$	$X[k] = X^*[-k]$
Symmetry (imag)	$\mathfrak{Re}\{\tilde{x}(t)\} = 0$	$X[k] = -X^*[-k]$
Parseval	$\frac{1}{T}\int_{-T/2}^{T/2} \lvert\tilde{x}(t)\rvert^2\, dt = \sum_{k=-\infty}^{\infty} \lvert X[k]\rvert^2$	

Properties of the Fourier Transform

Property	***Time domain***	***Frequency domain***
Linearity	$a x_1(t) + b x_2(t)$	$a X_1(j\omega) + b X_2(j\omega)$
Duality	$X(jt)$	$2\pi x(-\omega)$
Differentiation	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{1}{j\omega} X(j\omega) + \pi X(j0)\delta(\omega)$
Time-shift	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Frequency-shift	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega) X_2(j\omega)$
Modulation	$x_1(t) x_2(t)$	$\frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$
Time-reversal	$x(-t)$	$X(-j\omega)$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Symmetry (real)	$\mathfrak{Im}\{x(t)\} = 0$	$X(j\omega) = X^*(-j\omega)$
Symmetry (imag)	$\mathfrak{Re}\{x(t)\} = 0$	$X(j\omega) = -X^*(-j\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{j\omega}{a}\right)$
Parseval (Rayleigh)	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\, d\omega$	

Simple Example

$$f(t) = 10 + \cos(\omega_0 t) + 3\cos(2\omega_0 t)$$

What is the equivelent function in $F(j\omega)$

Delay functions

- $H\{x[n]\} = x[n-1]$
- $H\{x[n]\} = x[n-0.5]$

How is a half sample delay performed in time domain?