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Introduction to Communications

Lecture 7: Correlation & Spectral Density

This lecture:

1. Correlation of Energy Signals
2. Energy Spectral Density
3. Correlation of Power Signals
4. Power Spectral Density
5. Properties of Correlation & Spectral Density

Ref: CCR pp. 124–135, Couch pp. 61–65.

Correlation of Energy Signals

Recall that a signal $x(t)$ is called an energy signal if $E\{x(t)\} < \infty$.

- We define the (cross-)correlation of two real-valued energy signals $x(t)$ and $y(t)$ as

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) y(t-\tau)\, dt.$$

- The parameter τ is termed the time difference or lag.

Correlation of Energy Signals

- The correlation measures how similar $x(t)$ and $y(t)$ are at lag τ.
  - A large value indicates a high degree of similarity.
- For complex-valued energy signals, we define correlation as

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) y^*(t-\tau)\, dt.$$

- Observe the similarity with convolution. In fact,

$$R_{xy}(\tau) = x(\tau) * y^*(-\tau).$$

Energy Cross-Spectral Density

Consider the Fourier transform of $R_{xy}(\tau)$ (taken over τ instead of the usual t).

- We call this the energy cross-spectral density $G_{xy}(f)$.
- From the convolution & time-reversal properties of the Fourier transform, we have

$$G_{xy}(f) = X(f)Y^*(f).$$

Autocorrelation

We can calculate the correlation of a signal with itself.

- This is autocorrelation which, for an energy signal, is

$$\mathrm{R_x}(\tau) = \mathrm{x}(\tau) * \mathrm{x}^*(-\tau).$$

Energy Spectral Density

It follows that we can similarly define the energy spectral density (ESD) $G_x(f)$ as the Fourier transform of $R_x(\tau)$ so that

$$E_x(f) = X(f)X^*(f) = |X(f)|^2 .$$

Energy Spectral Density (2)

- The name is explained by application of Parseval's theorem:

$$\mathrm{E}\{\mathrm{x}(\mathrm{t})\} = \int_{-\infty}^{\infty} |\mathrm{X}(\mathrm{f})|^2 \, \mathrm{df} = \int_{-\infty}^{\infty} \mathrm{G_x}(\mathrm{f}) \, \mathrm{df}.$$

- Observe also that

$$\mathrm{E}\{\mathrm{x}(\mathrm{t})\} = \int_{\infty}^{\infty} |\mathrm{x}(\mathrm{t})|^2 \, \mathrm{dt} = \mathrm{R_x}(0).$$

Correlation of Power Signals

Recall that a signal $x(t)$ is called a power signal if $P\{x(t)\} < \infty$.

- For power signals, we require a slightly different definition of correlation since the definition for energy signals doesn't converge.

Correlation of Power Signals (2)

- Instead, we define the cross-correlation of two power signals $x(t)$ and $y(t)$ in terms of (asymptotic) time averages:

$$R_{xy}(\tau) = \langle x(t) y^*(t-\tau) \rangle$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) y^*(t-\tau)\, dt.$$

- Similarly, autocorrelation of a power signal $x(t)$ is defined as

$$R_x(\tau) = \langle x(t) x^*(t-\tau) \rangle .$$

Power Spectral Density

A signal with non-zero power is (by definition!) not square integrable, so we can't compute its Fourier transform.

- Hence, the spectral density defined for energy signals is not applicable to power signals either, so we take another approach.

Power Spectral Density (2)

- Defining $x_T(t) = \Pi(t/T)x(t)$, the power can be computed as

$$P\{x(t)\} = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |x_T(t)|^2 \, dt.$$

- By Parseval's equality, it follows that

$$P\{x(t)\} = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |X_T(f)|^2 \, df.$$

Power Spectral Density(3)

- Exchanging limits & integration above, we define the power spectral density (PSD) of $x(t)$ as

$$G_x(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{T}. \tag{1}$$

The Wiener-Khintchine Theorem

The Wiener-Khintchine theorem states that

$$\mathrm{R_x}(\tau) \xleftrightarrow{\mathrm{FT}} \mathrm{G_x}(\mathrm{f}).$$

- This is the same relationship that holds for energy signals

We can calculate the PSD in two different ways:

1. Direct method, using (1) on the previous slide.
2. Indirect method, by calculating the autocorrelation first, then FT.

Power Cross-Spectral Density

Similarly, it's possible to define a power cross-spectral density of two power signals $x(t)$ and $y(t)$ as

$$G_{xy}(f) = \lim_{T \to \infty} \frac{X_T(f) Y_T^*(f)}{T}.$$

- The Wiener-Khintchine theorem can be generalised, so that

$$R_{xy}(\tau) \overset{FT}{\longleftrightarrow} G_{xy}(f).$$

Properties

... of Correlation & Spectral Density

The following additional properties apply to correlation and spectral density:

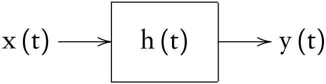
1. $\mathrm{R_x}(0) = \mathrm{E}\{\mathrm{x}(\mathrm{t})\}$ for energy signals and $\mathrm{R_x}(0) = \mathrm{P}\{\mathrm{x}(\mathrm{t})\}$ for power signals.
2. $\left|\mathrm{R_{xy}}(\tau)\right|^2 \leq \mathrm{E}\{\mathrm{x}(\mathrm{t})\}\mathrm{E}\{\mathrm{y}(\mathrm{t})\}$ for energy signals and $\left|\mathrm{R_{xy}}(\tau)\right|^2 \leq \mathrm{P}\{\mathrm{x}(\mathrm{t})\}\mathrm{P}\{\mathrm{y}(\mathrm{t})\}$ for power signals.
3. $|\mathrm{R_x}(\tau)| \leq \mathrm{R_x}(0)$.

Properties (2)

4. $R_{xy}(-\tau) = R^*_{yx}(\tau)$.
5. $R_x(-\tau) = R^*_x(\tau)$.
6. $G_x(f)$ is real.
7. $G_x(f) \geq 0$.
8. When $x(t)$ is real, $G_x(f)$ is even.

LTI Systems

There are some additional properties that apply when $x(t)$ is the input to an LTI system and $y(t)$ is the corresponding output.



$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

1. $R_{xy}(\tau) = h(\tau) * R_x(\tau)$.
2. $R_y(\tau) = h^*(-\tau) * h(\tau) * R_x(\tau)$.
3. $G_{xy}(f) = H(f) G_x(f)$.
4. $G_y(f) = |H(f)|^2 G_x(f)$.