Trades and Defining Sets:
Theoretical and Computational Results

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Statement of originality

The work presented in this thesis is, to the best of my knowledge and belief, original, except as acknowledged in the text. No part of this work has been previously submitted for a degree at this, or any other, university.

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NOTE: Some sections of the work were undertaken jointly with Brenton D. Gray; this work comprises Sections 3.1–3.7, Sections 4.1–4.6, Theorem 4.42, and Appendices C and D.

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Abstract

Given a particular combinatorial structure, there may be many distinct objects having this structure. When investigating these, two natural questions to ask are:

- ▷ Given two objects, where and how do they differ?
- ▷ How much of an individual object is necessary to uniquely identify it?

These two questions are obviously related, with the first leading to the concept of a *trade*, and the second to that of a *defining set*. In this thesis we study trades and defining sets, in the context of t-(v, k, λ) *designs*. In our enquiries, we make use of both theoretical and computational techniques.

We investigate the *spectrum* of trades, and prove an extant conjecture regarding this. Our results also suggest a more general version of this conjecture. A t-(v, k, λ) design where $\lambda = 1$ is called a *Steiner* design, and the related trades are called Steiner trades. In the case $t = 2$, we establish the spectrum of Steiner trades for each value of k, except for a finite number of values in each case.

The connection between trades and defining sets is used to obtain some new theoretical results on defining sets of designs, and is exploited throughout the thesis. We also consider the collections of all trades and all defining sets in a design.

A *simple* design is one which is a set, as opposed to a multiset. We present an algorithm to enumerate all the trades in simple designs. For non-simple designs we introduce the concept of a *discriminating set*, and present an algorithm to enumerate these. Output from these algorithms was used to investigate the trades and defining sets of a number of designs, and some new results were obtained.

Given part of a design, its *completions* are all those designs that contain it. An existing algorithm to complete partial designs is examined, and a heuristic yielding a much improved algorithm for Steiner designs is discussed. This completion routine was used to investigate a number of designs, and new information on the size and distribution of their defining sets was obtained.

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Chapter 1

Introduction

In this thesis, we study *trades* and *defining sets*, and the relationship between them. These are general concepts, applicable to many combinatorial structures. We restrict our attention to t-(v,k,λ) *designs*. (The survey article [69] considers these concepts applied to Latin squares and graphs, in addition to designs. The article [85] considers vertex colourings of graphs and Latin rectangles.) Throughout our investigations we make use of both theoretical and computational techniques.

In this introductory chapter, we start by defining the problems we address in Section 1.1, and present some motivational material in Section 1.2. We only include enough material in this chapter to define our problems. The required background material on designs, trades, and defining sets, including definitions of various technical terms, is given in the next chapter.

In Section 1.3 the organisation of this thesis is detailed, while Section 1.4 outlines the conventions used throughout. Details of the computational environment used are given in Section 1.5.

1.1 Problems addressed

Definition 1.1: *Let V be a v-set, and suppose that $\mathcal{B}$ is a collection of k-subsets of V, with the property that each t-subset of V is in exactly λ of the elements of $\mathcal{B}$. Then the ordered pair $(V,\mathcal{B})$ is called a t-(v,k,λ)* ***design****. The elements of V are called* ***points****, and the elements of $\mathcal{B}$* ***blocks****.*

The number of blocks, $|\mathcal{B}|$, in a design is denoted by b, and each point appears in exactly r blocks. The **parameters** of a design are t, v, k, λ, b and r. A generic design is denoted by $D = (V,\mathcal{B})$. Throughout, when we speak of a **design** or a t**-design** we will always mean a t-(v,k,λ) design.

Definition 1.2: *The set of distinct blocks in $\mathcal{B}$ is called the* ***support*** *of the design, and the number of distinct blocks is called the* ***support size****, denoted b^*. If $b^* = b$,* *then the design is said to be* ***simple****.*

DEFINITION 1.3: *A design with $\lambda = 1$ is called a* ***Steiner*** *design; such designs are necessarily simple.*

Suppose that $D_1 = (V, \mathcal{B}_1)$ and $D_2 = (V, \mathcal{B}_2)$ are two t-(v, k, λ) designs on the same point set, and that $\mathcal{B}_1 \neq \mathcal{B}_2$. Let $\mathcal{I} = \mathcal{B}_1 \cap \mathcal{B}_2$ and consider $\mathcal{B}_1 \setminus \mathcal{I}$ and $\mathcal{B}_2 \setminus \mathcal{I}$. These two collections are disjoint and non-empty, and each t-subset of V occurs the same number of times in each collection. We think of them as representing the 'difference' between D_1 and D_2.

NOTATION: We follow the standard convention of writing pairs of collections such as $\mathcal{B}_1 \setminus \mathcal{I}$ and $\mathcal{B}_2 \setminus \mathcal{I}$ in the form $\mathcal{B}_1 \setminus \mathcal{I} - \mathcal{B}_2 \setminus \mathcal{I}$. Here '$-$' does not represent the set-difference binary operation, which is always represented by '$\setminus$'. We think of the blocks of $\mathcal{B}_1 \setminus \mathcal{I}$ as being labelled '$+$' and those of $\mathcal{B}_2 \setminus \mathcal{I}$ as being labelled '$-$'.

DEFINITION 1.4: *Let V be a v-set and T_1, T_2 be collections of m k-subsets of V. We say that T_1 and T_2 are t-**balanced** if each t-subset of V is contained in the same number of blocks of T_1 and of T_2. If T_1 and T_2 are disjoint and t-balanced, then $T = T_1 - T_2$ is said to be a (v, k, t)* ***trade****. If $T_1 = T_2 = \emptyset$, the trade is said to be* ***void*** *or* ***null****. We call m the* ***volume*** *of the trade.*

If $T = T_1 - T_2$ is a (v, k, t) trade, we often refer to the single collection T_1 as a trade. If $D = (V, \mathcal{B})$ is a t-(v, k, λ) design with $T_1 \subseteq \mathcal{B}$, then the design is said to **contain** the trade. If $D_2 = (V, \mathcal{B}_2)$ is disjoint from D, then D is itself a trade; that is, $\mathcal{B} - \mathcal{B}_2$ is a trade. All designs contain the void trade, which is the difference between the design and itself. Obviously, any trade in a t-(v, k, λ) design is a (v, k, t) trade. However, not every (v, k, t) trade need appear in a particular t-(v, k, λ) design, even allowing for the t-subset multiplicity; that is, the value of λ.

DEFINITION 1.5: *Suppose that the collection T_1 is a non-void (v, k, t) trade. If, for all proper subcollections R, $\emptyset \subset R \subset T_1$, there does not exist a disjoint collection Q such that R and Q are t-balanced, then the trade is said to be* ***minimal****.*

Let $\mathcal{T}$ be the collection of all non-void trades in a design. If we delete from $\mathcal{T}$ all trades that properly contain another trade in $\mathcal{T}$, then the collection is said to be **minimised**. The trades in such a minimised collection are obviously minimal. If $\mathcal{T}$ is a collection of only some of the trades in a design, or an arbitrary collection of trades, then we can also minimise it. In this case, the resulting trades need not be minimal. However we may speak loosely, and call these trades minimal, with the qualification "with reference to the collection $\mathcal{T}$" being understood.

Definition 1.6: *Suppose that $T = T_1 - T_2$ is a (v, k, t) trade. If neither T_1 nor T_2 contains any repeated blocks, then the trade is said to be* ***simple****. If no t-subset of V occurs more than once in T_1, then T is said to be* ***Steiner****.*

Remark: Note that the Steiner property is well-defined, since T_1 and T_2 are t-balanced. It is possible for T_1 to be simple and T_2 to be non-simple; consider, for example, the non-simple $(6, 2, 1)$ trade with $T_1 = +\{\{1,3\},\{1,4\},\{2,5\},\{2,6\}\}$ and $T_2 = -\{\{1,2\},\{1,2\},\{3,4\},\{5,6\}\}$.

Although we have defined trades in the context of designs, they are interesting in their own right. In this thesis we are interested in the *spectrum* – that is, the possible volumes – and 'structure' of general, simple and Steiner trades. We also consider what structure the collection of minimal trades in a design might have.

The number of designs with given parameters and the number of blocks in a design can both be very large. To uniquely specify a particular design, must we list all b of its blocks, or will a smaller number suffice?

Definition 1.7: *Given a t-(v, k, λ) design $D = (V, \mathcal{B})$, a subset $S \subseteq \mathcal{B}$ of the blocks of D that occurs in no other t-(v, k, λ) design is called a* ***defining set*** *of D. If no proper subset of S is a defining set, then S is called a* ***minimal*** *defining set of D. If S is a minimal defining set of D and all other defining sets of D have at least $|S|$ blocks, then S is called a* ***smallest*** *defining set of D.*

Trivially, the answer to our question is yes, a smaller number will suffice. Any $b-1$ blocks of D uniquely define it, since the final block is *forced*. So all designs have defining sets that are proper subsets of $\mathcal{B}$, and thus non-trivial smallest defining sets.

Notation: We use dD to denote a defining set of D, and d_mD (resp. d_sD) to denote a minimal (resp. smallest) defining set. We use $|d_sD|$ to denote the number of blocks in a smallest defining set of D.

In this thesis we are interested in $|d_sD|$, the range of values which $|d_mD|$ can take, and how many *different* smallest, or minimal, defining sets there are in a design. We also consider what structure the collection of smallest/minimal defining sets of a design might have.

Given a collection $\mathcal{C}$ of k-subsets of V, any design D which contains $\mathcal{C}$ is said to be a **completion** of $\mathcal{C}$, and $\mathcal{C}$ is said to **complete** to D. If $\mathcal{C}$ is in only one design D, and is thus a defining set, $\mathcal{C}$ is said to **complete uniquely**, and D is said to be the **unique completion** of $\mathcal{C}$. Irrespective of whether or not $\mathcal{C}$ completes to any design, we call $\mathcal{C}$ a **partial design**, or simply a **partial**.

When investigating defining sets, an efficient programme for completing partials is an obvious desideratum. In this thesis we are interested in completion programmes; in particular, we are concerned with improving their efficiency and the uses to which they can be put.

1.2 Motivation

As well as being interesting in their own right, trades have many uses in the theory of designs. They can be used to construct designs with different support sizes and to construct *non-isomorphic* designs from a given design [61, 62, 63, 84]. They are closely related to the design intersection problem [7]. Trades are also frequently used implicitly, in a variety of guises: for example, (n, t)-partitionable sets are used in [2] in halving the *full* design. If $n = 2$, then (n, t)-partitionable sets are trades.

Designs, trades and *signed* or *integral* designs can all be subsumed in the same algebraic setting, and consideration of this may perhaps lead to a general method of constructing designs [65, 77, 59].

Obviously, when constructing defining sets of a design D, the differences between D and the other designs with the same parameters must be taken into account. So trades and defining sets are intimately linked. This close connection is illustrated by the following two results.

Lemma 1.8: ([50]) *Every defining set of a design $D = (V, \mathcal{B})$ contains a block of every possible trade $T_1 \subseteq \mathcal{B}$.* □

Lemma 1.9: ([50]) *If $D = (V, \mathcal{B})$ is a design and $S \subseteq \mathcal{B}$ contains a block of every minimal trade in D, then S is a defining set of D.* □

Throughout, we are interested in how we can make use of the connection between trades and defining sets. Many of our results use knowledge in one area to derive results in the other.

One potential application of defining sets is to *secret sharing schemes* or *access* *schemes*. Here, each participant in the scheme is given some partial information or *share*. Various subsets of participants can combine their shares to form a *key*, thereby gaining access to the secret (the key itself can constitute the secret). Designs could perhaps be used as keys, with the participants' shares being some of the blocks of the design. Knowledge about a design's defining sets would be very useful in allocating the participants' shares. See [15, 44, 109], and the references therein, for more details.

Although developed to assist the study of defining sets, completion routines are useful in other areas. They can be used to study *embeddings*; that is, can a t-(v, k, λ) design be completed to a t-(u, k, λ) design, for some $u > v$? They can also be used to study *extensions*; that is, can a t-(v, k, λ) design, with a new point added to each block, be completed to a $(t + 1)$-$(v + 1, k + 1, \lambda)$ design.

1.3 Organisation of thesis

In the next chapter we present the background material on designs we require in the remainder of the thesis, and review the existing results on trades and defining sets. We open the thesis proper with a study of trades. In Chapter 3 we investigate the spectrum and structure of general (v, k, t) trades, and do the same for Steiner trades in Chapter 4.

In Chapter 5 we move on to consider trades and defining sets in designs. The connections between these are used to obtain some new theoretical results on defining sets of designs. We also briefly discuss the collections of trades and defining sets in designs.

The remaining six chapters can be loosely grouped into pairs, with the first member of each pair describing a particular computational technique and the second member detailing some of the results obtained via this technique. In Chapter 6 we present an algorithm to enumerate all the trades in simple designs, and we use this algorithm in Chapter 7 to investigate several parameter sets. In Chapters 8 and 9 we present a 'generalisation' of trades to *multisets*, and show how these can be used to investigate non-simple designs. In Chapter 10 we examine an existing algorithm to complete partial designs, and present an improved algorithm for Steiner designs. In Chapter 11 we use this algorithm to investigate several designs.

Appendix A collects the various symbols, notation, abbreviations, acronyms and terms used throughout the thesis into a series of tables for ease of reference. Entries in these tables include a section reference (for example, §8.1) or a page reference (for example, p. 9) to the point of definition or first use.

In Appendix B we summarise all the results of which we are aware on defining sets of specific designs. Tables B.1, B.2 and B.3 contain all known values of $|d_sD|$. The designs are arranged in lexicographic order of (t, v, k, λ); we also give r, b, and n, the number of *non-isomorphic* designs. In the $|d_sD|$ column, we use the notation s^m to mean that m of the designs have $|d_sD| = s$. If the reference column includes a section reference, then the result, or part thereof, is original work and is presented in the section listed.

We do not list in Table B.1 et al. cases where all the blocks of a design are forced. These *trivial* designs have $|d_sD| = 0$; for example, the 2-$(4, 2, 1)$ and 2-$(4, 3, 2)$ designs mentioned in [46, 47]. We also do not list *complementary* designs, such as the 4-$(11, 6, 3)$ design tabulated in [111]; $|d_sD|$ for this can be found using Lemma 1.8 of [47] and the value for the 4-$(11, 5, 1)$ design, as discussed in [93, p. 100].

Table B.4 summarises the results obtained for various generalisations of the concept of a defining set, while Table B.5 summarises what is known regarding the distribution of smallest and minimal defining sets in designs.

In Table B.6 of Appendix B, we summarise results regarding trades in particular designs. This summary includes only results obtained as part of our work; results in the literature are included only if they are discussed in the text.

Throughout the thesis, many of our results are presented in tabular form. Where convenient, these are incorporated into the relevant chapters. If there is a large amount of tabular material associated with a chapter, the bulk of the material is collected into an appendix at the end of the thesis. We also relegate some tangential material and longer proofs to appendices. The appendices from Appendix C onwards contain all of this material.

During the course of our work we verified many of the results in the literature. We may not explicitly note these verified results. We do, however, always note where our results extend existing results or, in one case (see §7.1.4), conflict with them.

1.4 Conventions

Throughout this thesis, V always stands for a v-set. The points of our trades and designs are normally drawn from V, which is taken to be either $\{0, \ldots, v-1\}$ or $\{1, \ldots, v\}$. For convenience, we sometimes use $a, \ldots, z$ to stand for $10, \ldots, 35$. We also use the notation $\underline{\cdot}$ or $\underline{\underline{\cdot}}$ to indicate an element distinct from those already used.

To avoid trivialities, we assume throughout that $0 < t < k < v$. We use $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) to stand for the greatest (resp. least) integer $\leq x$ (resp. $\geq x$).

Our trades and designs may be multisets; however, we generally use set notation, trusting to context to disambiguate this. Only in Chapter 8, where we explicity manipulate multisets, do we depart from this convention. We use the terms *collection* and *family* freely as synonyms of set. When writing blocks and sets of blocks, we normally omit separating commas and braces where possible. In particular, we write blocks as square-free monomials; thus $\{012, abc\}$ stands for $\{\{0, 1, 2\}, \{a, b, c\}\}$.

When terms are defined, we use a bold font thus: **trade**. Where terms are used before their definition or left undefined, we use an emphasised font thus: *complementary*. We also use this font for terms that we do not use but mention en passant. We indicate the first occurrence of terms used imprecisely by single quotes thus: ‘difference.’

The names of our programmes, third-party programmes, and operating system utilities are indicated using a typewriter font thus: `opbdp`, `prof`. We occasionally use this monospaced typewriter font to represent points, where we wish to neatly align tables or to distinguish between points from V and integers.

1.5 Computational environment

All the computational work described was performed in a Unix environment, with the programmes being written in C. The GNU project’s C compiler, `gcc` [28], was used throughout. For information about any of the standard Unix utilities and tools mentioned, or programming in C, consult any good reference text, such as [70, 106], or the on-line manuals.

Extensive use was made of the utility `nauty` [91], which is a set of procedures for determining the *automorphism* group of a vertex-coloured graph. It provides a set of generators, the size of the group, and its orbits. It can also produce a canonicallylabelled isomorph of the graph, for isomorphism testing. If a design is represented by a bipartite graph, then the automorphism group of this graph is the same as that of the design. The algorithm is fully described in [90], while [79] contains a tutorial-style introduction to the techniques employed.

REMARK: The output of `nauty` has to be interpreted with care if the design is non-simple, since a permutation of repeated blocks is considered to be an automorphism.

Occasional use was made of the computer language `magma` [9]. In particular, it was used to investigate the *transitivity* of various automorphism groups.

Given a list of trades, or trade-like structures, in a design D, a lower bound for $|d_sD|$ can be obtained by solving a certain Boolean, or pseudo-Boolean, optimisation problem. The utility `opbdp` [5] was used to find optimal solutions for these problems.

Existing utilities for completing partials (`complete` [18]) and finding defining sets (`bds` [21]) were used as references, for comparison with our programmes.

Apart from the utilities noted above, all computations were performed using programmes developed by the author. A variety of systems was used. These included: Sun or PC-based servers and workstations in the Department of Computer Science & Electrical Engineering and the Department of Mathematics at The University of Queensland; the Queensland Parallel Supercomputer Facility's IBM SP2 supercomputer; and the Silicon Graphics' Power Challenge array supercomputer at The University of Queensland's High Performance Computing Unit.

Chapter 2

Background

In this chapter we present the background material necessary for the remainder of the thesis. There is a wealth of material available concerning designs; we content ourselves with presenting only that required. As has been noted, trades are used frequently, often implicitly, throughout design theory. We review the basic theory that we need. For defining sets the material is more limited; we review the greater part of the extant literature.

2.1 Designs

The material in this section is drawn in the main from standard texts. For further material on t-(v, k, λ) designs, other types of designs, and design and combinatorial theory see, for example, [4, 6, 58, 112] or a handbook such as [14].

Standard counting arguments show that the parameters of a design are related by

$$rv = bk \quad \text{and} \quad \lambda\binom{v}{t} = b\binom{k}{t}.$$

Further, each u-subset of V appears in exactly

$$\lambda_u = \lambda\binom{v-u}{t-u} \Big/ \binom{k-u}{t-u}$$

blocks of the design, for $0 \leq u \leq t$. Note that $\lambda_0 = b$, $\lambda_1 = r$ and $\lambda_t = \lambda$. For a t-(v, k, λ) design to exist, it is necessary that all of these λ_u be integers. A set of parameters for which this is the case is called **admissible**. Admissibility does not imply existence. It is an open, and difficult, problem in design theory to establish sufficient conditions for the existence of designs.

Let S_v denote the symmetric group of permutations on a v-set. Two designs, $D_1 = (V, \mathcal{B}_1)$ and $D_2 = (V, \mathcal{B}_2)$, are said to be **isomorphic** if there exists $\rho \in S_v$ such that $\rho\mathcal{B}_1 = \mathcal{B}_2$. (For a block $B = \{x_1, \ldots, x_k\}$ we define $\rho B = \{\rho x_1, \ldots, \rho x_k\}$. We also define $\rho\mathcal{B} = \{\rho B : B \in \mathcal{B}\}$.) If no such ρ exists, then D_1 and D_2 are **non-isomorphic**. We sometimes use **different** as a synonym for non-isomorphic. If

$\mathcal{B}_1 \neq \mathcal{B}_2$, then D_1 and D_2 are said to be **distinct**. Distinct designs may or may not be isomorphic.

NOTATION: We normally use n to denote the number of non-isomorphic t-(v, k, λ) designs with a given set of parameters, and work with $\mathcal{D} = \{D_0, \ldots, D_{n-1}\}$, a **transversal** of the designs. That is, any t-(v, k, λ) design is isomorphic to precisely one D_i, $0 \leq i \leq n-1$. We use N (resp. N_i) to represent the total number of distinct designs (resp. number of distinct designs isomorphic to D_i), and $\mathcal{D}^*$ (resp. D_i^*) to represent the set of all distinct designs (resp. distinct designs isomorphic to D_i). If $n = 1$ we often say that the design is **unique**, even if $N > 1$.

Given a design D, if $\rho \in S_v$ is such that $\rho\mathcal{B} = \mathcal{B}$, then ρ is called an **automorphism** of D. The set of all automorphisms of D is denoted by $\mathrm{aut}(D)$, and is a subgroup of S_v; that is, $\mathrm{aut}(D) \subseteq S_v$. If the only automorphism of D is the identity permutation, then $\mathrm{aut}(D)$ is said to be **trivial**, and D is said to be **rigid**. If D is a t-(v, k, λ) design, then the number of distinct designs isomorphic to D is given by $|S_v|/|\mathrm{aut}(D)| = v!/|\mathrm{aut}(D)|$.

The automorphism group of a design partitions both the set of blocks and the set of points into **orbits**. Two blocks, or two points, are in the same orbit if and only if there is an element of $\mathrm{aut}(D)$ that maps one to the other. If there is a single orbit of blocks (resp. points), the design is said to be **block-transitive** (resp. **point-transitive**). If any set of s blocks (resp. s points) can be mapped to any other set of s blocks (resp. s points), then the design is said to be **block s-transitive** (resp. **point s-transitive**). We use **transitive** as a shorthand for block-transitive.

If there exists $\rho \in \mathrm{aut}(D)$ of order $v = |V|$, then the design is said to be **cyclic**. In a cyclic design, the set of points V can be identified with $\mathbb{Z}_v$, the residue group of integers modulo v. In this case, we have $\rho : i \mapsto i + 1 \pmod{v}$. The cyclic automorphism ρ partitions $\mathcal{B}$ into block orbits. Orbits of size v are said to be **full**, and orbits that are not full are called **short**. An arbitrary block from each orbit can be chosen as the **starter block**, and the design can be generated by **developing** the set of starter blocks.

For a starter block $B = \{x_1, \ldots, x_k\}$ define $\triangle B = \{x_i - x_j : i, j = 1, \ldots, k; i \neq j\}$, and for a family of starter blocks $\mathcal{F}$ define $\triangle\mathcal{F} = \bigcup_{B \in \mathcal{F}} \triangle B$. In a cyclic 2-$(v, k, 1)$ design, either all the orbits are full or there is precisely one short orbit. If all the orbits are full, then $\triangle\mathcal{F} = \mathbb{Z}_v \setminus \{0\}$, and $\mathcal{F}$ is called a $(v, k, 1)$ **difference set** or

difference family, depending on whether $|\mathcal{F}| = 1$ or $|\mathcal{F}| > 1$.

Example 2.1: The 2-$(7, 3, 1)$ design is unique, and an example is $D = (V, \mathcal{B})$, where $V = \{0, \ldots, 6\}$ and $\mathcal{B} = \{013, 124, 235, 346, 450, 561, 602\}$. So D is cyclic, with a single full orbit, and can be developed from, say, starter block 013. That is, $\{0, 1, 3\}$ is a $(7, 3, 1)$ difference set. $\mathrm{Aut}(D)$ has order 168, is 2-transitive on the points and on the blocks, and there are exactly 30 distinct designs. □

Remark: A transitive design need not be cyclic, nor need a cyclic design be transitive.

Given t, k and v, the collection of all k-subsets of V obviously forms a simple t-$\left(v, k, \binom{v-t}{k-t}\right)$ design. This design is called the **full** design, and we define $\lambda^* = \binom{v-t}{k-t}$. The minimum λ for which the parameter set is admissible is denoted by λ_* and, in any t-(v, k, λ) design, $\lambda = m\lambda_*$ for some $m > 0$. Of course, given a simple t-(v, k, λ) design D with $\lambda < \lambda^*$, if we subtract the blocks of D from the full design then we obtain a t-$(v, k, \lambda^* - \lambda)$ design.

Given a t-(v, k, λ) design D, the t-$(v, k, m\lambda)$ design formed by taking $m > 1$ copies of D is called a **multiple** design. If $t < k < v - t$, we can complement the blocks of D with respect to V to obtain a t-$\left(v, v-k, \lambda\binom{v-k}{t}/\binom{k}{t}\right)$ design, called the **complementary** design. If a design is equal to its complement, it is said to be **self-complementary**. Given this complementation process, we normally work with the design that has $k \leq v/2$. This, together with the condition $0 < t < k$, also ensures that our designs are not **trivial**; that is, they are not empty, nor are they necessarily the full design or a multiple thereof.

Fisher's Inequality states that $b \geq v$ in a 2-design. If $b = v$ in a 2-(v, k, λ) design, then $k = r$, and the design is called **symmetric**. In a **linked** design, any pair of blocks intersect in the same number of points; this number is called the **linkage**. Symmetric designs are necessarily linked, with linkage λ. In a **quasi-symmetric** design each pair of blocks intersect in one of two possible numbers of points.

If the blocks of a design can be partitioned into classes each of which contains each point exactly α times, the design is said to be **α-resolvable**. A 1-resolvable design is said to be **resolvable** and, in this case, the classes are called **parallel** classes.

Example 2.2: The 2-$(7, 3, 1)$ design in Example 2.1 is symmetric, with linkage 1. Consider the 2-$(9, 3, 1)$ design whose twelve blocks are given by the rows, columns, forward diagonals, and backward diagonals of the array

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}\,.$$

This design is resolvable, with the parallel classes being the rows, columns, forward diagonals, and backward diagonals. The design is also quasi-symmetric, with blocks in the same class being disjoint, and those in different classes intersecting in one point. □

Suppose that $D = (V, \mathcal{B})$ is a symmetric 2-(v, k, λ) design, and let B be any block in $\mathcal{B}$. Then the collection of intersections of B with the remaining blocks of $\mathcal{B}$ forms a 2-$(k, \lambda, \lambda - 1)$ design, called the **derived** design. If B and the derived design are removed from $\mathcal{B}$, then a 2-$(v - k, k - \lambda, \lambda)$ design, called the **block-residual** design, is obtained.

Suppose that $D = (V, \mathcal{B})$ is a t-(v, k, λ) design, and let x be any point in V. The **restriction** of D on x is the $(t-1)$-$(v-1, k-1, \lambda)$ design formed by taking all the blocks that contain the point x and deleting x. The **point-residual** design is the $(t-1)$-$(v-1, k, \lambda(v-k)/(k-t+1))$ design formed by taking the blocks of D that do not contain the point x. A t-(v, k, λ) design has v possible restrictions. If these are all isomorphic, the design is said to be **homogeneous**; if not, **inhomogeneous**.

Suppose that $D = (V, \mathcal{B})$ is a t-(v, k, λ) design, and let $\underline{x}$ be a new point not in V. It may be possible to **extend** D to a $(t+1)$-$(v+1, k+1, \lambda)$ design, called an **extension** of D, by adding $\underline{x}$ to all the blocks of $\mathcal{B}$ and then completing this set of blocks. Extension and restriction are each other's inverse; however extension is not always possible, unlike restriction. If the set of blocks added when completing an extension is the complement of the original blocks (with respect to $V \cup \{\underline{x}\}$), then the process is called **extension by complementation**. An extended design formed by complementation is necessarily self-complementary.

Example 2.3: The 2-$(7, 3, 1)$ design can be extended to a 3-$(8, 4, 1)$ design. The extension is by complementation, and so the 3-design is self-complementary. The 3-design is unique and is homogeneous, since the 2-$(7, 3, 1)$ design is unique. □

If D is a t-design and $S \subseteq \mathcal{B}$ is also a t-design, then S is a **subdesign** of D. Subdesigns have the same values of t and k, but may have smaller v and/or λ. If D is a t-(v, k, λ) design with a t-(u, k, λ) subdesign S for some $0 < u < v$, then S is said to be **embedded** in D. If D is a t-(v, k, λ) design with a t-(v, k, κ) subdesign for some

$0 < \kappa < \lambda$, then D is said to be **decomposable**; otherwise it is **indecomposable**. (The terms *reducible* and *irreducible* are sometimes used in the literature.)

For given t, v and k the number of indecomposable designs is finite [26]. This number is not known, in general. However, when all the indecomposable designs are known, all t-(v, k, λ) designs can be constructed [57]. Note that a design may have more than one decomposition into indecomposable designs. In fact, designs exist where such decompositions give different partitionings of λ; see [13] for a 2-$(9, 3, 3)$ design where λ can be decomposed as $\{1, 1, 1\}$ or $\{2, 1\}$. We produce another example in Chapter 9.

The set of all subspaces of the $(n+1)$-dimensional vector space over the Galois field $GF[q]$ is called the **projective geometry** of **dimension** n over $GF[q]$ (notation, $PG(n, q)$). The subspaces of dimension 1, 2, 3 and n are called **points**, **lines**, **planes** and **hyperplanes** respectively. Various designs can be constructed from the incidence relationship between subspaces of $PG(n, q)$. In particular, the points and lines of $PG(2, q)$ yield the **projective plane** of **order** q, a 2-$(q^2 + q + 1, q + 1, 1)$ design. Projective planes are cyclic, symmetric, and have linkage 1.

Given $PG(n, q)$, if a line and all its points is removed we obtain the **affine geometry** of **dimension** n over $GF[q]$ (notation, $AG(n, q)$). The points and lines of $AG(2, q)$ yield the **affine plane** of **order** q, a 2-$(q^2, q, 1)$ design. Affine planes are block-residual designs of the projective planes, and are resolvable and quasi-symmetric, with linkages 0 and 1. An extension of an affine plane to a 3-$(q^2 + 1, q + 1, 1)$ design is an **inversive plane**.

EXAMPLE 2.4: The 2-$(7, 3, 1)$ design is the projective plane of order 2, and is called the **Fano** plane. The 2-$(9, 3, 1)$ design is the affine plane of order 3, and is the block-residual of the projective plane of order 3. The 2-$(9, 3, 1)$ design has three extensions to the unique 3-$(10, 4, 1)$ inversive plane. □

The 2-$(v, 3, 1)$ and 3-$(v, 4, 1)$ Steiner designs are called **Steiner triple systems** and **Steiner quadruple systems** of **order** v (abbreviated $STS(v)$ and $SQS(v)$) respectively. They exist for all $v \equiv 1, 3 \pmod 6$ and $v \equiv 2, 4 \pmod 6$ respectively. A symmetric 2-$(4m - 1, 2m - 1, m - 1)$ design is called a **Hadamard** design. Any Hadamard design can be extended by complementation to a **Hadamard 3-design**, a 3-$(4m, 2m, m - 1)$ design.

Given a particular group G, we can sometimes construct a design D with $G =$

$\mathrm{aut}(D)$ or $G \subset \mathrm{aut}(D)$. In particular, for the small Mathieu groups M_{11}, M_{12} and the large Mathieu groups M_{22}, M_{23}, M_{24}, we can construct 4-$(11,5,1)$, 5-$(12,6,1)$ and 3-$(22,6,1)$, 4-$(23,7,1)$, 5-$(24,8,1)$ designs having these groups as their automorphism groups. These designs are called the **small/large Mathieu** designs. (They are often referred to in the literature as the *Witt* designs.)

Example 2.5: The 2-$(7,3,1)$ and the 3-$(8,4,1)$ designs given previously are the unique $STS(7)$ and $SQS(8)$. The Fano plane is a Hadamard design, and the $SQS(8)$ is a Hadamard 3-design. The small and large Mathieu designs are successive extensions of the projective planes of order 3 and 4 respectively. □

2.2 Trades

In Chapters 3 and 4 we derive many new results on trades. In these chapters we make use of existing results, often recasting them into a form more suitable for our needs. In this section, we review some basic trade theory. We start our review with a fundamental observation.

Lemma 2.6: ([61, 67]) *Let $T = T_1 - T_2$ be a (v,k,t) trade.*
(1) *For all $0 < t' < t$, T_1 and T_2 are t'-balanced, and T is a (v,k,t') trade.*
(2) *For all $v' > v$, T is a (v',k,t) trade.* □

So, if $T = T_1 - T_2$ is a (v,k,t) trade, then there may exist elements of V which occur in no block of T. The set of elements of V contained in a set of blocks X is called the **foundation** of X, denoted by $F(X)$. Obviously $F(T_1) = F(T_2)$, and so we define $F(T) = F(T_1)$. We set $f(T) = |F(T)|$, so $f(T) \leq v = |V|$. We also let $m(T) = |T_1| = |T_2|$ denote the volume of the trade. We sometimes use **t-trade** for (v,k,t) trade.

Example 2.7: $T = T_1 - T_2 = +135 + 146 + 236 + 245 - 136 - 145 - 235 - 246$ is a $(6,3,2)$ trade, with $F(T) = \{1,2,3,4,5,6\}$, $f(T) = 6$ and $m(T) = 4$. By Lemma 2.6 T is also, for example, a $(7,3,1)$ trade. □

It was shown in [67] that, if T is a non-void (v,k,t) trade, then $m(T) \geq 2^t$ and $v \geq f(T) \geq k+t+1$. Trades of volume 2^t always exist, and are called **basic** trades.

Theorem 2.8: ([67, 75]) *If $v \geq k+t+1$ and $k \geq t+1$, then there exists a (v,k,t) trade of volume 2^t. Such a trade has the following form:*

$$T = T_1 - T_2 = S_0(S_1 - S_2)(S_3 - S_4)\cdots(S_{2t+1} - S_{2t+2}),$$

where $S_i \subset V$ *for* $i = 0, \ldots, 2t+2$, $S_i \cap S_j = \emptyset$ *for* $i \neq j$, $|S_{2i-1}| = |S_{2i}| \geq 1$ *for* $i = 1, \ldots, t+1$, *and* $|S_0| + \sum_{i=1}^{t+1} |S_{2i}| = k$. □

A $(k+t+1, k, t)$ basic trade is called a **smallest** trade. These trades were called minimal in [67, 75], but we avoid this usage here, to prevent confusion with the minimal trades of Definition 1.5.

EXAMPLE 2.9: If we put $S_0 = \emptyset$ and $S_i = \{i\}$, $i = 1, \ldots, 6$, in Theorem 2.8, then we obtain the (smallest) trade of Example 2.7. □

In a basic trade, we call S_0 the **tail** of the trade, and note that if the tail is non-empty then it occurs in all blocks of the trade. If $k = t+1$, then the tail is necessarily empty, as in Example 2.9. Smallest trades are unique and, amongst the (v, k, t) basic trades, have maximum-sized tails. The smallest $(6, 3, 2)$ trade of Example 2.7 is called a **Pasch** trade, and a set of blocks isomorphic to T_1 of this trade is called a **Pasch configuration**.

LEMMA 2.10: ([61]) *Let* V *be a* v*-set, and suppose* $T = T_1 - T_2$ *and* $R = R_1 - R_2$ *are* (v, k, t) *trades, with* $F(T)$, $F(R) \subseteq V$. *Then* $T + R = (T_1 + R_1) - (T_2 + R_2)$ *is a* (v, k, t) *trade.* □

REMARK: The definition of a trade $T = T_1 - T_2$ requires that $T_1 \cap T_2 = \emptyset$. So if in Lemma 2.10 the two *halves* of $T + R$ have blocks in common, these 'cancel' in the sum; that is, they are deleted. Note also that $F(T + R) \subseteq F(T) \cup F(R)$.

Hence the (v, k, t) trades are closed under addition, with the null trade being the additive identity. The inverse of $T = T_1 - T_2$ is $T_2 - T_1$, and addition is obviously associative and commutative. So the collection of all (v, k, t) trades is an (infinite) abelian group, and thus a $\mathbb{Z}$-module.

The (v, k, t) basic trades are a spanning set for this module – see [33, 35, 67] – so any trade can be written as a sum of basic trades. In fact, this $\mathbb{Z}$-module has dimension $\binom{v}{k} - \binom{v}{t}$, and several authors have described bases for it; see the review in [76].

DEFINITION 2.11: *Let* $T = T_1 - T_2$ *be a trade. Then* $B \in T_1 \cup T_2$ *is the* ***starting block*** *of* T *if* B *is first, in lexicographic order, among all the blocks of* $T_1 \cup T_2$. *An ordered set of trades* $\{T^1, \ldots, T^n\}$, *with respective starting blocks* $\{B^1, \ldots, B^n\}$, *is* ***semitriangular*** *if* $i < j$ *implies that* B^i *precedes* B^j, *in lexicographic order.*

The construction of a semitriangular basis of smallest trades was described in [73]; see also [111, 76]. We now describe this construction, which was the one used in some of our computer-based searches for trades.

In a smallest trade $|S_0| = k - t - 1$, and all other S_i are singletons. We set $V = \{1, \ldots, v\}$, and take $(b_1 - c_1) \cdots (b_{t+1} - c_{t+1}) b_{t+2} \cdots b_k$ as a smallest trade, where $b_i, c_i \in V$ and all the b_i, c_i are distinct. Since T and $-T$ have the same starting block, we can assume that the starting block is $b_1 \cdots b_k$, and that $b_1 < \cdots < b_{t+1}$, $b_{t+2} < \cdots < b_k$, and $b_i < c_i$ for $1 \leq i \leq t+1$.

REMARK: The construction was developed under the assumption that $b_{t+1} < b_{t+2}$.

LEMMA 2.12: ([73]) *The block* $B = \{b_1, \ldots, b_k\}$, *in which* $b_1 < \cdots < b_k$, *is the starting block of a smallest* (v, k, t) *trade if and only if:*
(1) $b_i \leq v - k - t + 2i - 2$, *for* $1 \leq i \leq t + 1$;
(2) $b_i \leq v - k + i$, *for* $t + 2 \leq i \leq k$. □

LEMMA 2.13: ([73]) *For* $k > t > 0$ *and* $v \geq k + t + 1$ *there are precisely* $\binom{v}{k} - \binom{v}{t}$ *starting blocks.* □

From each starting block $B = \{b_1, \ldots, b_k\}$, a smallest trade is constructed by setting

$$c_{t+1} = \min(\{b_{t+1} + 1, \ldots, v\} \setminus \{b_{t+2}, \ldots, b_k\}),$$

and then setting

$$c_i = \min(\{b_i + 1, \ldots, v\} \setminus \{c_{i+1}, \ldots, c_{t+1}, b_{i+1}, \ldots, b_k\}),$$

for $i = t, \ldots, 1$ (in that order). Since the collection of these trades is semitriangular, and there are precisely $\binom{v}{k} - \binom{v}{t}$ of them, they obviously form a basis for the module.

EXAMPLE 2.14: In Figure 2.1 the semitriangular basis, of dimension 28, for the $\mathbb{Z}$-module of $(8, 5, 2)$ trades obtained by this construction is given. The trade polynomials, ordered by starting block, are given down the left-hand column. Along the top, the 56 5-subsets of V are given in lexicographic order, with the blocks written vertically. Each row of '`+`'s and '`-`'s represents the trade to its left. The '`.`'s are purely a visual aid. The rows (i.e., basic trades) can be thought of as 56-element vectors, with entries drawn from $\{-1, 0, +1\}$, □

Recall that a non-void trade has volume at least 2^t. This condition is only sufficient when $t = 1$. For $0 \leq i \leq t$, define $s_i = 2^t + 2^{t-1} + \cdots + 2^{t-i} = 2^{t+1} - 2^{t-i}$. A trade of volume $s_0 = 2^t$ is a basic trade. In [86], (v, k, t) trades of volume s_i, $i = 1, 2, 3$ and $t \geq i$, were constructed. It was further shown that trades of volume s, where $s_0 < s < s_1$, do not exist. This led to the following conjecture, which we investigate in the next chapter.

FIGURE 2.1: A semitriangular basis for $(8,5,2)$ trades

```
                  11111111111111111111111111111111111222222222222222333334
                  22222222222222222222333333333344445333333333344445444455
                  33333333334444445556444444555655566444444555655566555666
                  44445556675556676677555667667766777555667667766777667777
                  56786787886787887888678788788878888678788788878888788888
(1-8)(2-7)(3-6)45 +.........-..........-........+......-........+.....+..-
(1-8)(2-7)(3-5)46 .+........-............-......+........-......+......+.-
(1-8)(2-6)(3-5)47 ..+........-...........-......+.........-......+.....+.-
(1-7)(2-6)(3-5)48 ...+........-...........-......+........-......+.....+.-
(1-8)(2-7)(3-4)56 ....+.....-...............-...+...........-...+.......+-
(1-8)(2-6)(3-4)57 .....+.....-..............-...+............-...+......+-
(1-7)(2-6)(3-4)58 ......+.....-..............-...+...........-...+......+-
(1-8)(2-5)(3-4)67 .......+.....-............-...+.............-...+.....+-
(1-7)(2-5)(3-4)68 ........+.....-............-...+............-...+.....+-
(1-6)(2-5)(3-4)78 .........+.....-............-...+...........-...+.....+-
(1-8)(2-3)(4-7)56 ..........+.....-...-.....+...................-..+.+..-.
(1-8)(2-3)(4-6)57 ...........+....-....-....+....................-.+..+.-.
(1-7)(2-3)(4-6)58 ............+....-....-....+...................-.+..+.-.
(1-8)(2-3)(4-5)67 .............+..-......-..+.....................-+...+-.
(1-7)(2-3)(4-5)68 ..............+..-......-..+....................-+...+-.
(1-6)(2-3)(4-5)78 ...............+..-......-..+...................-+...+-.
(1-4)(2-3)(5-8)67 ................+..-......-..+...............-..+.+..-..
(1-4)(2-3)(5-7)68 .................+.-.......-.+................-.+..+.-..
(1-4)(2-3)(5-6)78 ..................+-........-+.................-+...+-..
(1-2)(3-8)(4-7)56 ....................+.....-....-..+-.....+....+..-......
(1-2)(3-8)(4-6)57 .....................+....-.....-.+.-....+.....+.-......
(1-2)(3-7)(4-6)58 ......................+....-....-.+..-....+....+.-......
(1-2)(3-8)(4-5)67 .......................+..-......-+...-..+......+-......
(1-2)(3-7)(4-5)68 ........................+..-.....-+....-..+.....+-......
(1-2)(3-6)(4-5)78 .........................+..-....-+.....-..+....+-......
(1-2)(3-4)(5-8)67 ..........................+..--..+.......-..++..-.......
(1-2)(3-4)(5-7)68 ...........................+.-.-.+........-.+.+.-.......
(1-2)(3-4)(5-6)78 ............................+-..-+.........-+..+-.......
```

CONJECTURE 2.15: ([86])

(1) *For all* $0 \leq i \leq t$, *there exists a* (v,k,t) *trade of volume* s_i.

(2) *For all* s, $s_i < s < s_{i+1}$, $0 \leq i \leq t-1$, *no* (v,k,t) *trade of volume* s *exists.*

If $T = T_1 - T_2$ is a trade, we call T_1 and T_2 the **halves** of the trade, and say that T_2 is a **trade mate** for T_1. Note that the halves of a trade need not be isomorphic. Further, given T_1, the collection T_2 need not be 'forced' (up to isomorphisms); that is, T_1 may have multiple non-isomorphic mates.

EXAMPLE 2.16: Recall, from the remark after Definition 1.6, the $(6,2,1)$ trade with $T_1 = \{13, 14, 25, 26\}$ and $T_2 = \{12, 12, 34, 56\}$. Here, T_1 and T_2 are obviously non-isomorphic. Given T_2, then its mate T_1 is forced, up to a permutation of $\{3,4,5,6\}$. However, T_1 also trades with $T_3 = \{12, 15, 23, 46\}$, which is different from T_2. Note that T_2 and T_3 are not trade mates, since they are not disjoint. $\square$

2.3 Defining sets

The concept of a defining set of a design was formally introduced in the series of articles [47, 46, 45], although an isolated earlier result concerning the 5-$(24, 8, 1)$ design from the large Mathieu group M_{24} was given in [16]. Two recent surveys are [110, 111]. Appendix B should be consulted for results regarding specific designs. Throughout the thesis, we use μ to stand for $|d_sD|/b$, the proportion of the blocks of a design in a smallest defining set.

The first thing to note is that the property of having a unique completion is invariant under permutations. This observation yields the following results.

LEMMA 2.17: ([47]) *Suppose that S is a defining set of D and $\rho \in \operatorname{aut}(D)$. Then ρS is a defining set of D, and $\operatorname{aut}(S) \subseteq \operatorname{aut}(D)$.* □

THEOREM 2.18: ([45]) *Suppose that S is a defining set of D. Then*

$$\operatorname{aut}(D) = \{\rho : \rho S \subseteq D, \rho \in S_v\}. \qquad \square$$

DEFINITION 2.19: ([45]) *A permutation $\rho \in S_v$ of the form (ij), $i \neq j \in V$, is called a* ***single transposition****. A* ***single-transposition-free*** *(STF) design D is one whose automorphism group does not contain any single transpositions.*

It was established in [47, 45, 93] that t-$(v, k, 1)$ and symmetric 2-(v, k, λ) designs are STF. The following results have been obtained for STF designs.

THEOREM 2.20: ([47, 45]) *Let D be an STF t-(v, k, λ) design, and suppose that S is a dD, with $|S| = s$. Then:*
(1) $f(S) \geq v - 1$;
(2) *if points $i \neq j$ each appear once only in S, they appear in separate blocks;*
(3) $s \geq 2(v-1)/(k^* + 1)$, *where* $k^* = \min(k, v-k)$;
(4) $2^{s-1} \geq \max(k, v-k)$;
(5) $\binom{s}{2} + s + 1 \geq v$, *if* $\lambda = 1$. □

THEOREM 2.21: ([45]) *Let S be a defining set of a simple STF design D, and denote the number of configurations of blocks of D isomorphic to S by $n(S : D)$. Then*

$$n(S : D) = \frac{|\operatorname{aut}(D)|}{|\operatorname{aut}(S)|}. \qquad \square$$

THEOREM 2.22: ([45]) *Let $\mathcal{D}$ be a transversal of the t-(v, k, λ) designs, all of which are simple and STF. Let $D \in \mathcal{D}$, and suppose that S is a configuration on v or $v - 1$ points such that:*

(1) *D has precisely $|\mathrm{aut}(D)|/|\mathrm{aut}(S)|$ subsets of blocks isomorphic to S;*

(2) *any design containing a subset of blocks isomorphic to S is isomorphic to D.*

Then S is a defining set for some design isomorphic to D. □

These results have been used as the basis for algorithms to find and catalogue (smallest) defining sets, or to improve the efficiency of such algorithms [48, 53, 51, 52, 17]. In [37] the concept of an automorphism group of a set of blocks $\mathcal{B}$ was extended to include those points in V but not in $F(\mathcal{B})$. This allowed analogues of Theorem 2.21 and 2.22 to be proved for all simple designs.

The bounds of Theorem 2.20 are generally weak, and only apply to STF designs. Various other bounds, including upper bounds, have been proved.

THEOREM 2.23: ([44]) *Suppose that D is a cyclic symmetric 2-(v, k, λ) design, and contains a trade of volume m. If $m \nmid v$, then $|d_sD| \geq v/m$.* □

THEOREM 2.24: ([44]) *Suppose that D is a t-$(v, k, 1)$ design, with b blocks. Then the size of a minimal defining set of D satisfies*

$$|d_mD| \;\leq\; b - \frac{2\binom{v}{t-1}}{\binom{k}{t-1}+1}. \qquad \square$$

LEMMA 2.25: ([44]) *Suppose that $D = (V, \mathcal{B})$ is a t-$(v, t+1, \lambda)$ design, then*

$$|d_sD| \;\leq\; \lambda_0 - \lambda_1 = b - r.$$

PROOF: Let $S \subseteq \mathcal{B}$ consist of all blocks of D that do not contain some point, say 0, so that $|S| = b - r$. Then any t-subset of V that does not appear in λ blocks of S must appear an appropriate number of times as a block with 0. So S completes uniquely. □

Various asymptotic results regarding smallest defining sets of infinite families have been proved, or conjectured.

THEOREM 2.26: ([38]) *Let D_d be the cyclic Hadamard design arising from the points and hyperplanes of $PG(d, 2)$, $d \geq 2$, and let μ_d be the proportion of blocks in a smallest defining set. Then $\mu_d \to 1$ as $d \to \infty$.* □

THEOREM 2.27: ([38]) *Let D_d^* be the block-residual design of D_d, $d \geq 3$, and let μ_d^* be the proportion of blocks in a smallest defining set. Then $\mu_d^* \to 1$ as $d \to \infty$.* □

THEOREM 2.28: ([38]) *Let L_d be the $STS(2^{d+1} - 1)$ obtained from the points and lines of $PG(d, 2)$, $d \geq 2$, and let η_d be the proportion of blocks in a smallest defining* *set. Then the sequence $\{\eta_d\}_{d=2}^{\infty}$ is non-decreasing and bounded above by 1; thus, it converges.* □

Remark: Although a family of minimal defining sets where $|d_m D|/b \to 1$ is known (see Theorem 2.29 below), the limiting value of η_d is not known. It is known that $\eta_3 = 16/35$ [93].

If $p \equiv 1 \pmod 4$ is a prime power, then a Hadamard 2-$(2p+1, p, \frac{1}{2}(p-1))$ design can be constructed using quadratic residues in $GF(p)$. Similarly, if $p \equiv 3 \pmod 4$, then a Hadamard 2-$(p, \frac{1}{2}(p-1), \frac{1}{4}(p-3))$ design can be constructed. It has been conjectured that a particular set of p (resp. $\frac{1}{2}(p-1)$) blocks from these designs is a defining set; so, in particular, $\mu < 1/2$ for both families. This conjecture has been proved for $p = 5, 7, 9, 11, 13, 17, 19, 23, 25, 27, 29, 43, 47, 59$ and 67 [105, 104, 80].

Remark: These conjectured families of designs with $\mu < 1/2$ should be contrasted with that of Theorem 2.26, where $\mu \to 1$.

Using particular sets of hyperplanes in projective and affine geometries, two infinite families of minimal defining sets were described in [31, 32]. Note that in both these cases $|d_m D|/b$, the proportion of blocks in the minimal defining set, tends to 1 as $d \to \infty$.

Theorem 2.29: ([31]) *Let D be the $STS(2^{d+1}-1)$ obtained from the points and lines of $PG(d,2)$, $d \geq 2$. Then D has a minimal defining set of size*

$$\frac{1}{6}((2^{d+1}-1)(2^{d+1}-2) - 3^{d+1} + 3).$$ □

Theorem 2.30: ([32]) *Let D be the $STS(3^d)$ obtained from the points and lines of $AG(d,3)$, $d \geq 2$. Then D has a minimal defining set of size*

$$\frac{1}{6}(3^d(3^d-1) - 7^d + 1).$$ □

Since the number of indecomposable designs is finite, the majority of t-(v,k,λ) designs are decomposable and the following result is helpful. Note that, since a design may have more than one decomposition, for the best lower bound we should take the maximum over all decompositions. We investigate this bound in more detail in Chapter 9.

Lemma 2.31: ([47]) *Suppose that the design D has a decomposition into the designs D_i, $i = 1, \ldots, m$. Then*

$$|d_s D| \geq \sum_{i=1}^{m} |d_s D_i|.$$ □

The relationships between a design and its complement, its extensions, and its derived or residual designs yield results concerning the sizes of smallest defining sets of all these designs. Many of the specific results quoted in Appendix B were obtained using these connections.

Lemma 2.32: ([47]) *If S is a defining set of D, then the complement of S is a defining set of the complement of D.* □

Lemma 2.33: ([95]) *If a t-design D has exactly one extension E to a $(t+1)$-design, then any defining set of D, when extended, is a defining set of E. Hence $|d_sE| \leq |d_sD|$.* □

Lemma 2.34: ([95]) *If D is a t-design, t even, and E is an extension by complementation of D, then $|d_sE| \geq |d_sD|$.* □

Theorem 2.35: ([47]) *If D is a 3-$(2n+2, n+1, \lambda)$ design, where all such designs are necessarily obtainable by extension by complementation, and D^x its restriction on x, then $|d_sD| = |d_sD^x|$.* □

Lemma 2.36: ([95]) *If D_1 and D_2 are t-designs, t even, with a common extension by complementation E, then $|d_sD_1| = |d_sD_2|$. Further, D_1 and D_2 have the same number of smallest defining sets.* □

Lemma 2.37: ([95]) *Suppose D is a t-design, t even, and the only extension of D is the extension by complementation E. If D has precisely n smallest defining sets, of size q, and E has precisely m smallest defining sets, then $n \leq m \leq n2^q$. The upper bound is attained if all designs with the same parameters as E are self-complementary.* □

Lemma 2.38: ([93]) *Let $D = (V, \mathcal{B})$ be a t-(v, k, λ) design, and suppose that $S_1 \subseteq \mathcal{B}$ completes to precisely m other t-(v, k, λ) designs $D_1, \ldots, D_m$. Let $S_2 \subseteq \mathcal{B} \setminus S_1$ be such that $S_2 \not\subseteq \mathcal{B}_i$, $i = 1, \ldots, m$. Then $S_1 \cup S_2$ is a defining set of D.* □

The previous lemma is a convenient source of 'small' defining sets, if a suitable set S_1 can be found. In particular, it can be used to prove a series of results regarding defining sets of residual designs [93].

Let $\mathcal{M}$ be the collection of all minimal defining sets of a design D. Given the various results and bounds on smallest/minimal defining sets, it is of interest to consider $\mathrm{spec}_m(D) = \{|M| : M \in \mathcal{M}\}$, the **defining spectrum** of D, see [44]. A defining spectrum $\mathrm{spec}_m(D)$ contains a **hole** if there exist $l < m < n$ such that

$l, n \in \mathrm{spec}_m(D)$ and $m \notin \mathrm{spec}_m(D)$. If D is the $STS(15)$ constructed from the points and lines of $PG(3,2)$, then it was shown in [39], using [93, 31, 44], that $\mathrm{spec}_m(D) = \{16, 17, 18, 19, 20, 21, 22\}$. We revisit the defining spectrum problem in Chapter 11, and obtain many new results.

In this thesis we consider defining sets of designs consisting of sets of blocks from the design; that is, **blockwise** defining sets. We do not consider defining sets containing incomplete blocks; that is, **pointwise** defining sets [17, 20, 19]. Nor do we consider defining sets for *large sets* or *overlarge sets* [17, 22, 82].

CHAPTER 3

General trades

Suppose that $T = T_1 - T_2$ is a (v, k, t) trade. Recall that not all elements of V need appear in a block of T. If, however, $f(T) = v$ then we introduce a new notation, and refer to T as a $[v, k, t]$ **trade**. Where we do not know, or have no interest in, the value of v, we speak of a (k, t) **trade** instead of a $[v, k, t]$ trade.

An obvious question to ask is, given parameters v, k and t, for which volumes does there exist a $[v, k, t]$ trade or a (k, t) trade? Accordingly, we make the following definitions.

DEFINITION 3.1: *The* ***spectrum*** *of all* $[v, k, t]$ *trades is*

$$\mathcal{S}[v,k,t] = \{m(T) : T \textit{ is a } [v,k,t] \textit{ trade}\} \cup \{0\}.$$

We also define the spectra

$$\mathcal{S}(k,t) = \bigcup_{v} \mathcal{S}[v,k,t] \quad \textit{and} \quad \mathcal{S}(t) = \bigcup_{k} \mathcal{S}(k,t).$$

Note that the void trade is a $[0, k, t]$ trade; we have defined $0 \in \mathcal{S}[v, k, t]$ for notational convenience. Clearly, $\mathcal{S}[v, k, t] \subseteq \mathcal{S}(k, t) \subseteq \mathcal{S}(t)$. We do not formally define the spectrum of (v, k, t) trades, $\mathcal{S}(v, k, t)$, since we do not study this herein.

EXAMPLE 3.2: Combining results from [7, 67, 86], it is known that $\mathcal{S}(1) = \{0, 2, 3, 4, \ldots\}$, $\mathcal{S}(2) = \{0, 4, 6, 7, 8, \ldots\}$ and $\mathcal{S}(3) = \{0, 8, 12, 14, 15, 16, \ldots\} \cup X$, where $X = \emptyset$ or $\{13\}$. □

In this chapter we are mainly interested in $[v, k, t]$ and (k, t) trades. We study their spectra and structure, using both theoretical techniques and computer-based searches. We prove Conjecture 2.15(1) and partially characterise the foundations of trades having volume s_i. We also prove, as a special case of Conjecture 2.15(2), that $2^t + 2^{t-1} + 1 \notin \mathcal{S}(t + 1, t)$, for $t \geq 3$. We raise a number of conjectures, including an extended version of Conjecture 2.15. The spectra $\mathcal{S}[v, k, 1]$ and $\mathcal{S}[v, 3, 2]$ are completely determined, and a variety of structural theorems are proved.

We consider general trades in this chapter, deferring to the next chapter our study of Steiner trades. We do not address the spectra of simple trades directly in these chapters, but note that many of our constructions can be applied to this problem. In passing, we also note that a simple $[v, k, t]$ trade can have volume at most $\binom{v}{k}/2$, and that constructing a simple $[v, k, t]$ trade of this volume is equivalent to halving the full design [60].

The work reported in Sections 3.1–3.7 of this chapter has been published in [42, 43].

3.1 A proof of Conjecture 2.15(1)

To illustrate one of our main techniques, we provide, in Theorem 3.4, a proof of Conjecture 2.15(1). (This result is also proved implicitly later in the chapter.) When adding basic trades, we show how we can manipulate the exact form of these trades to control the number of cancellations and hence the resulting volume. Later in this chapter, we also show how this allows us to achieve a range of foundations. All the trades we construct in this section are simple.

Recall, from Lemma 2.10, that we can add trades. If, in this lemma, $F(T) \cap F(R) = \emptyset$, then $m(T + R) = m(T) + m(R)$ and $f(T + R) = f(T) + f(R)$. Obviously, if we take c copies of a $[v, k, t]$ trade of volume m, we obtain a $[v, k, t]$ trade of volume cm. When adding different trades, we have to consider the possibility of blocks cancelling.

Lemma 3.3: *Suppose $T^a = T_1^a - T_2^a$ and $T^b = T_1^b - T_2^b$ are (k, t) trades. Then $T = T^a + T^b = T_1^a + T_1^b - T_2^a - T_2^b$ is a (k, t) trade of volume*

$$m(T^a) + m(T^b) - |T_1^a \cap T_2^b| - |T_2^a \cap T_1^b|.$$

Proof: The volume of T equals $m(T^a) + m(T^b)$ minus the number of blocks in $T_1^a \cap T_2^b$ and $T_2^a \cap T_1^b$. □

Theorem 3.4: *For all $0 \le i \le t$, there exists a (k, t) trade of volume s_i.*

Proof: Let T and R be basic (k, t) trades of the form

$$\begin{aligned} T &= T_1 - T_2 = S_0(S_1 - S_2)(S_3 - S_4)\dots(S_{2t+1} - S_{2t+2}), \\ R &= R_1 - R_2 = -S_0(S_1 - S_2)\dots \\ &\qquad \dots(S_{2t-2i-1} - S_{2t-2i})(\underline{S}_{2t-2i+1} - S_{2t-2i+2})\dots(\underline{S}_{2t+1} - S_{2t+2}), \end{aligned}$$

where $0 \le i \le t$, and $\underline{S}_j$ is chosen so that $\underline{S}_j \cap S_l = \emptyset$ for $j = 2t - 2i + 1, 2t - 2i + 3, \dots, 2t + 1$ and $l = 0, 1, \dots, 2t + 2$.

That $T+R$ is a (k,t) trade follows from Lemma 2.10. It remains to find $m(T+R)$. There are two cases to consider, $i=t$ and $i<t$, depending on whether 2^{t-i} is odd or even. When $i=t$, the only block common to T and R is $S_0S_2\cdots S_{2t+2}$. This block is in $T_1 \cap R_2$ or $T_2 \cap R_1$, hence $m(T+R) = m(T)+m(R)-1 = 2^{t+1}-1$. When $0 \leq i \leq t-1$, $|T_1 \cap R_2| = |T_2 \cap R_1| = 2^{t-i-1}$. Therefore,

$$m(T+R) = m(T)+m(R) - 2.2^{t-i-1} = 2^{t+1} - 2^{t-i},$$

as required. $\square$

Example 3.5: Let $T=(1-2)(3-4)(5-6)$, and choose $R=-(1-2)(\underline{3}-4)(\underline{5}-6)$. Then

$$\begin{aligned} T+R &= +135+236+245+1\underline{3}6+14\underline{5}+2\underline{35} \\ &\quad -136-145-235-1\underline{35}-2\underline{3}6-24\underline{5} \end{aligned}$$

is an $(8,3,2)$ trade with $m(T+R)=6$ and $f(T+R)=8$. $\square$

Corollary 3.6: *For $s \geq (2t-1)2^t$ there exists a (v,k,t) trade of volume s.*

Proof: Let $0 \leq a \leq 2^t-1$. Then a has a unique binary representation $\sum_{i=1}^{t} a_i 2^{t-i}$, where $a_i \in \{0,1\}$. Therefore, if $s^* = t.2^{t+1}-a$, where $0 \leq a \leq 2^t-1$, then

$$s^* = \sum_{i=1}^{t} a_i s_i + \sum_{i=1}^{t} (1-a_i)2^{t+1}.$$

Any integer $s > (2t-1)2^t$ can be written in the form $s = b.2^t + s^*$, where $s^* = t.2^{t+1}-a$, with $0 \leq a \leq 2^t-1$, and $b \geq 0$. Using disjoint foundations, choose b basic trades, for each $a_i = 1$ a trade of volume s_i, and for each $a_i=0$ two trades of volume 2^t (or a trade of volume 2^{t+1} as in [86], if a smaller foundation size is required). Now use Lemma 2.10 to construct a trade of volume s as required.

For $s=(2t-1)2^t$, choose $(2t-1)$ basic trades on disjoint foundations, and use Lemma 2.10 to construct the required trade. $\square$

Remark: Note that $(2t-1)2^t = 2(t-1)2^t + 2^t$, and so a trade of this volume can be constructed by adding a basic trade and $t-1$ pairs of basic trades. By introducing cancellations between the members of each pair, it is possible to reduce the lower bound to $2(t-1)2^t+2$.

3.2 Preliminaries

In this section we present some results and constructions which we will require later in the chapter. Some of these are new, and some are drawn from the literature but have been recast in terms of $[v, k, t]$ trades, in order to make the foundations explicit.

Notation: For a collection A of l-subsets and an element $x \notin F(A)$, xA denotes the collection of $(l+1)$-subsets formed by adjoining x to each of the l-subsets in A.

Lemma 3.7: *If $m \in \mathcal{S}[v, k, t]$, then $m \in \mathcal{S}[v+1, k+1, t]$.*

Proof: Let $T = T_1 - T_2$ be a $[v, k, t]$ trade of volume m, and choose $x \notin F(T)$. Then $xT_1 - xT_2$ is easily seen to be a $[v+1, k+1, t]$ trade of volume m, using Lemma 2.6 for the t-subsets containing x. $\square$

It follows that $\mathcal{S}(k, t) \subseteq \mathcal{S}(k+i, t)$ for all $i > 0$. So, to prove that a (k, t) trade of volume m exists for all $k > t$, it suffices to prove the case $k = t+1$.

Example 3.8: Consider the (Steiner) trade T of Example 2.7, and choose $x \notin F(T)$. Then a (non-Steiner simple) $[7, 4, 2]$ trade is given by $+135x + 146x + 236x + 245x - 136x - 145x - 235x - 246x$. $\square$

Lemma 3.9: *Suppose T^a and T^b are $[v_a, k, t]$ and $[v_b, k, t]$ trades of volumes m_a and m_b respectively. Then $m_a + m_b \in \mathcal{S}[v_a + v_b - i, k, t]$ for all $0 \leq i \leq k-1$.*

Proof: Relabel $F(T^a)$ and $F(T^b)$ so that $|F(T^a) \cap F(T^b)| = i$. Then $T^a + T^b$ is a $[v_a + v_b - i, k, t]$ trade. Since $i < k$, T^a and T^b cannot contain any block in common, so $m(T^a + T^b) = m_a + m_b$. $\square$

It is a consequence of this result that $\mathcal{S}(k, t)$ is closed under addition. Thus, when investigating $\mathcal{S}(k, t)$ or $\mathcal{S}(t)$, we need only find a contiguous range of 2^t volumes for which trades exist, since we can then add basic trades to obtain all larger volumes.

Lemma 3.10: *Suppose $T^a = T_1^a - T_2^a$ and $T^b = T_1^b - T_2^b$ are $[v_a, k, t]$ and $[v_b, k, t]$ trades of volumes m_a and m_b respectively, such that $F(T^b) \subseteq F(T^a)$ and $T_1^a \cap T_2^b = T_2^a \cap T_1^b = \emptyset$. Then $m_a + cm_b \in \mathcal{S}[v_a + i, k, t]$ for all $c > 0$ and $0 \leq i \leq cv_b$.*

Proof: Simply add c copies of T^b to T^a, relabelling the foundations of the T^b as necessary to yield the required overall foundation. $\square$

Lemma 3.11: *Suppose $T^a = T_1^a - T_2^a$ and $T^b = T_1^b - T_2^b$ are $[v_a, k, t]$ and $[v_b, k, t]$ trades of volumes m_a and m_b respectively, and that one of T_1^b, T_2^b, T_1^a or T_2^a contains some block once only. Then $m_a + m_b - 1 \in \mathcal{S}[v_a + v_b - k, k, t]$.*

Proof: Suppose, without loss of generality, that block B occurs in T_1^a exactly once. Relabel the elements of any block C of T_2^b so that $C = B$ and relabel the elements of $F(T^b) \setminus B$ so that $F(T^a) \cap (F(T^b) \setminus B) = \emptyset$. Now B is the only block common to T^a and T^b, and hence $m(T^a + T^b) = m_a + m_b - 1$. $\square$

Remark: Since basic trades are simple and always exist, and since 2^t and $2^{t+1} - 1$ are coprime, this result implies that the complement of $\mathcal{S}(k, t)$ is finite – although it does not give a bound as strong as that of Corollary 3.6.

Lemma 3.12: ([86]) *If* $m \in \mathcal{S}[v, k, t]$ *then* $2m \in \mathcal{S}[v + 2, k + 1, t + 1]$.

Proof: Let $T = T_1 - T_2$ be a $[v, k, t]$ trade of volume m, and choose $x, y \notin F(T)$ with $x \neq y$. Then $(xT_1 \cup yT_2) - (yT_1 \cup xT_2)$ is a $[v + 2, k + 1, t + 1]$ trade of volume $2m$. $\square$

Remark: Lemmas 3.11 and 3.12 can be used to provide an alternative proof of Theorem 3.4. All the even s_i can be obtained by inducting on t, while s_t can be obtained using two basic trades and Lemma 3.11.

Definition 3.13: *Suppose* A *is a collection of* k*-subsets and* $x \in F(A)$. *Then* $R_x(A) = \{a \setminus \{x\} : a \in A, x \in a\}$ *is the* ***restriction*** *on* x *of* A. *We say that* $r_x(A) = |R_x(A)|$ *is the* ***multiplicity*** *of* x *in* A. *We call* $L_x(A) = \{a : a \in A, x \notin a\}$ *the* ***leave*** *of* x *in* A.

Suppose that $T = T_1 - T_2$ is a (k, t) trade of volume m and that $x \in F(T)$. Note that Lemma 2.6 implies that $r_x(T_1) = r_x(T_2)$; so we define $r_x(T) = r_x(T_1)$, and use r_x when no confusion arises. It was shown in [67] that $R_x(T_1) - R_x(T_2)$ is a $(k - 1, t - 1)$ trade of volume $r_x(T_1)$ and that $L_x(T_1) - L_x(T_2)$ is a $(k, t - 1)$ trade of volume $m - r_x$.

3.3 General results regarding foundations

In this section we fully determine the possible foundation sizes of trades of volume $r2^t$, $r > 0$. We also investigate when $\mathcal{S}[v, k, t]$ contains s_i. We start by giving an inequality relating the foundation size and volume of a trade.

Lemma 3.14: *Suppose that* $T = T_1 - T_2$ *is a* $[v, k, t]$ *trade of volume* m. *Then* $r_x(T) \geq 2^{t-1}$ *for all* $x \in F(T)$, *and so* $v2^{t-1} \leq mk$.

Proof: For $t = 1$, $r_x(T) \geq 1$, by definition. For $t > 1$, $R_x(T_1) - R_x(T_2)$ is a non-void $(k - 1, t - 1)$ trade, and so must have volume at least 2^{t-1}. Thus $r_x(T) \geq 2^{t-1}$ for all $x \in F(T)$, and the result follows. $\square$

Recall that $v \geq k+t+1$ in a non-void $[v,k,t]$ trade. We now show that this, together with Lemma 3.14, yields necessary and sufficient conditions for the existence of $[v,k,t]$ trades of volume $r2^t$, $r > 0$. Trades of these volumes can be constructed by adding r (non-cancelling) basic trades. We first characterise the foundations of basic trades.

LEMMA 3.15: *A $[v,k,t]$ trade of volume 2^t exists if and only if $k+t+1 \leq v \leq 2k$.*

PROOF: In Theorem 2.8,

$$\sum_{i=1}^{t+1} |S_{2i}| = \sum_{i=1}^{t+1} |S_{2i-1}| = k - |S_0|.$$

Thus,

$$f(T) = 2(k - |S_0|) + |S_0| = 2k - |S_0|.$$

Now $0 \leq |S_0| \leq k - (t+1)$ and we can always choose $|S_0|$ to be any value in this range, so the result follows. □

THEOREM 3.16: *For $r > 0$, there exists a $[v,k,t]$ trade of volume $r2^t$ if and only if $k+t+1 \leq v \leq 2rk$.*

PROOF: That $k+t+1 \leq v \leq 2rk$ is necessary follows from $f(T) \geq k+t+1$ and Lemma 3.14. For $k+t+1 \leq v \leq 2k$, take r copies of a basic trade with foundation v, which exists by Lemma 3.15. For $2k < v \leq 2rk$, apply Lemma 3.10 with $T^a = T^b$ both basic trades of foundation $2k$. □

The following result shows that $v \geq k+t+1$ and Lemma 3.14 are not sufficient for existence, in general. In particular, as $s_t = 2^{t+1} - 1$ is odd, it shows that $[2t+2, t+1, t]$ trades of volume s_t do not exist. Note that this result, along with Theorem 3.16, also shows that both $\mathcal{S}[2t+2, t+1, t]$ and its complement are infinite.

THEOREM 3.17: *Let T be a $[2t+2, t+1, t]$ trade. Then $m(T)$ is even.*

PROOF: T is also a $(2t+2, t+1, t)$ trade and thus we can assume that $T = \sum_{i=1}^{r} T^i$, for some $r > 0$, where the T^i are $(2t+2, t+1, t)$ basic trades (not necessarily distinct). If $m(T) = r2^t$ there is nothing to prove. If $m(T) < r2^t$, then there must exist $T^a = T_1^a - T_2^a$ and $T^b = T_1^b - T_2^b$, $1 \leq a < b \leq r$, such that either $T_1^a \cap T_2^b \neq \emptyset$ or $T_1^b \cap T_2^a \neq \emptyset$.

Since $v = k+t+1 = 2t+2$, in Theorem 2.8 $S_0 = \emptyset$ and all the S_i, $i \neq 0$, are singleton sets. So we can assume, without loss of generality, that $T^a = (1-2)(3-4)\cdots((2t+1)-(2t+2))$ and that $B = \{1, 3, \ldots, 2t+1\} \in T_1^a \cap T_2^b$. Since $B \in T_2^b$, we can write T^b in the form $-(1-\sigma(2))(3-\sigma(4))\cdots((2t+1)-\sigma(2t+2))$, where σ is a permutation of $\{2,4,\ldots,2t+2\}$.

Thus the block $B^* = \{2,4,\ldots,2t+2\}$ must be in both T^a and T^b. If t is even, then $B^* \in T_2^a$ and $B^* \in T_1^b$, while if t is odd, then $B^* \in T_1^a$ and $B^* \in T_2^b$. In either case, both B and B^* cancel in $T^a + T^b$. That is, blocks cancel in pairs, and the result follows. □

We now investigate trades with volume s_i. For a $[v_i,k,t]$ trade of volume s_i, $1 \le i \le t$, since $v_i \ge k+t+1$ and by Lemma 3.14, we must have $k+t+1 \le v_i \le s_i k/2^{t-1} = 4k - k/2^{i-1}$. Our first construction shows when the minimum foundation can be achieved.

Lemma 3.18: *For $1 \le i \le t$, $s_i \in \mathcal{S}[k+t+1,k,t]$ if and only if $(k,i) \neq (t+1,t)$.*

Proof: Given Theorem 3.17, since s_t is odd, we need only demonstrate existence.

For volume s_i, $1 \le i \le t-1$, and $k \ge t+1$, consider the two $[k+t+1,k,t]$ basic trades

$$S_0(1-2)(3-4)(5-6)\cdots((2t{+}1)-(2t{+}2)),$$
$$(-1^i)S_0(1-(2i{+}4))(2-3)\cdots((2i{+}2)-(2i{+}3))$$
$$((2i{+}5)-(2i{+}6))\cdots((2t{+}1)-(2t{+}2)),$$

where S_0 is a set of $k-t-1$ elements disjoint from $\{1,\ldots,2t{+}2\}$. These trades have the same foundation, and when summed they cancel in the 2^{t-i-1} blocks containing $S_0 13\cdots(2i{+}3)$ and the 2^{t-i-1} blocks containing $S_0 24\cdots(2i{+}4)$. So the sum is a $[k+t+1,k,t]$ trade of volume $2.2^t - 2.2^{t-i-1} = 2^{t+1} - 2^{t-i} = s_i$.

For volume s_t and $k \ge t+2$, consider the two $[k+t+1,k,t]$ basic trades

$$S_0 0(1-2)(3-4)(5-6)\cdots((2t{+}1)-(2t{+}2)),$$
$$-S_0 1(3-2)(5-4)(7-6)\cdots((2t{+}1)-2t)(0-(2t{+}2)),$$

where S_0 is a set of $k-t-2$ elements disjoint from $\{0,\ldots,2t{+}2\}$. These trades have the same foundation, and when summed the block $S_0 013\cdots(2t{+}1)$ cancels. □

Recall that S_0 is called the tail of a basic trade. If $T = T_1 - T_2$ is a basic trade, then any element of $F(T)$ not in the tail appears in exactly 2^{t-1} blocks in each of T_1 and T_2. In the constructions that follow we add two basic trades $T^a = T_1^a - T_2^a$ and $T^b = T_1^b - T_2^b$ with $F(T^a) \neq F(T^b)$ and cancel less than 2^{t-1} blocks from each of T_1^a, T_2^a, T_1^b and T_2^b. So $F(T^a + T^b)$ will always equal $F(T^a) \cup F(T^b)$.

We now consider the volume s_t case for $v_i \ge k+t+2$.

Lemma 3.19: *For all $k+t+2 \le v \le 3k$, $s_t \in \mathcal{S}[v,k,t]$.*

Proof: For $k+t+2 \le v \le k+2t+2$ consider the two basic trades

$$S_0(1-2)(3-4)(5-6)\cdots((2t{+}1)-(2t{+}2)),$$
$$-S_0(1-(2t{+}3))(3-2)(5-4)\cdots((2t{+}1)-2t),$$

where S_0 is a set of $k-t-1$ elements disjoint from $\{1,\ldots,2t+3\}$. When added, these trades cancel in the single block $S_0 13\cdots(2t+1)$, and yield a $[k+t+2,k,t]$ trade of volume s_t.

If we now replace 2 in the second trade by the new element $2t+4$, we obtain a $[k+t+3,k,t]$ trade. We can continue replacing the even elements in the second trade in this manner, obtaining all foundations up to $k+2t+2$.

For $k+2t+2 \le v \le 3k$, the result follows directly from Lemmas 3.15 and 3.11. □

We now consider the s_i, $1 \le i \le t-1$, for $v_i \ge k+t+2$.

Lemma 3.20: *For all $1 \le i \le t-1$ and $k+t+2 \le v \le 3k-(t-i)$, $s_i \in \mathcal{S}[v,k,t]$.*

Proof: Consider the two basic trades

$$S_0(S_1-S_2)(S_3-S_4)\cdots(S_{2t+1}-S_{2t+2}),$$
$$-S_0(S_1-S_2)\cdots(S_{2(t-i)-1}-S_{2(t-i)})(S_{2(t-i)+1}-\underline{S}_{2(t-i)+2})\cdots(S_{2t+1}-\underline{S}_{2t+2}),$$

These cancel in the 2^{t-i} blocks containing $S_0S_{2(t-i)+1}S_{2(t-i)+3}\cdots S_{2t+1}$, yielding a (k,t) trade of the required volume. We will achieve the required range of foundations by manipulating the $\underline{S}_y$.

(1) $2k+1+i \le v \le 3k-(t-i)$: Suppose that $|S_{2t+1}| = |S_{2t+2}| = |\underline{S}_{2t+2}| = k-t$, so $|S_0| = 0$ and all other S_x and $\underline{S}_y$ are singletons. Then $F(T^a+T^b) = 3k-(t-i)$. If we now reduce $|S_{2t+1}|$, $|S_{2t+2}|$ and $|\underline{S}_{2t+2}|$ by one, and increase $|S_1|$ and $|S_2|$ by one, then $F(T^a+T^b) = 3k-(t-i)-1$. We can continue reducing $|S_{2t+1}|$, $|S_{2t+2}|$ and $|\underline{S}_{2t+2}|$ and increasing $|S_1|$ and $|S_2|$ in this manner until $|S_{2t+1}| = |S_{2t+2}| = |\underline{S}_{2t+2}| = 1$, when $F(T^a+T^b) = 2k+1+i$.

(2) $k+t+2+i \le v \le 2k+1+i$: Suppose that $|S_0| = k-t-1$, with all other S_x and $\underline{S}_y$ being singleton sets. Then $F(T^a+T^b) = k+t+2+i$. If we now reduce $|S_0|$ by one, and increase $|S_1|$ and $|S_2|$ by one, $F(T^a+T^b) = k+t+3+i$. We can continue in this manner until $S_0 = \emptyset$, when $F(T^a+T^b) = 2k+1+i$.

(3) $k+t+2 \le v \le k+t+1+i$: Suppose we replace $\underline{S}_{2(t-i)+2}$ by $S_{2(t-i)+4}$. Then the number of cancellations, and hence the volume, remains unchanged. However,

in parts (1) and (2) these sets are singletons, so the foundations decrease by one. So we obtain the range $k+t+2+i-1 \leq v \leq 3k-(t-i+1)$. In general, for $1 \leq j \leq i$, we can replace the j singleton sets $\underline{S}_{2(t-i)+2}, \ldots, \underline{S}_{2(t-i)+2j}$ by the j singleton sets $S_{2(t-i)+4}, \ldots, S_{2(t-i)+2j+2}$, to obtain the range $k+t+2+i-j \leq v \leq 3k-(t-i+j)$. When $j=i$ this provides $v=k+t+2$, and when $j=1$ this provides $v=k+t+1+i$. Since the j ranges overlap, this provides all the required foundations. □

The results in this section on trades of volumes s_i prove the following theorem. We make a conjecture regarding the remaining possible foundations in Section 3.7.

THEOREM 3.21: *For all $1 \leq i \leq t$ and $k+t+1 \leq v_i \leq 3k-(t-i)$ there exists a $[v_i, k, t]$ trade of volume s_i, except that no $[2t+2, t+1, t]$ trade of volume s_t exists.* □

3.4 Proof of a special case of Conjecture 2.15(2)

As noted earlier, Conjecture 2.15(2) has been proved for volumes s, $s_0 < s < s_1$. The smallest unsettled volume is thus s_1+1, for $t \geq 3$, and we now prove this in the special case $k=t+1$. To do this, we require the following definition and structural lemma.

DEFINITION 3.22: *Let $T=T_1-T_2$ be a (k,t) trade, and suppose that $\emptyset \neq S_1 \subset T_1$ and $\emptyset \neq S_2 \subset T_2$. If S_1-S_2 is a (k,t) trade, then it is called a **subtrade** of T. Note that, in this case, $(T_1 \backslash S_1)-(T_2 \backslash S_2)$ is also a subtrade of T.*

LEMMA 3.23: *Suppose that $T=T_1-T_2$ is a $(t+1,t)$ trade of volume $r+2^{t-1}$, $t>1$ and $r>0$, and there exists $x \in F(T)$ with $r_x(T)=r$. Then T contains a subtrade of volume 2^t or 2^t+2^{t-1}.*

PROOF: Recall Definition 3.13, and its following remarks. Since $L_x(T)$ is a $(t+1, t-1)$ trade of volume 2^{t-1}, then $L_x(T)$ is $(t-1)$-balanced, but need not be t-balanced. However, T is t-balanced, so any t-subset in $L_x(T_2)$ that is not in $L_x(T_1)$ must occur as a block in $R_x(T_1)$, and similarly for the t-subsets in $L_x(T_1)$. Call this set of t-subsets in $R_x(T_1)$ (resp. $R_x(T_2)$) S_1 (resp. S_2). Obviously, $S_1 \cap S_2 = \emptyset$.

Now $L_x(T_1) \cup S_1$ and $L_x(T_2) \cup S_2$ are t-balanced. We will show that $|S_1|$ $(=|S_2|)$ is either 2^{t-1} or 2^t, and that S_1 and S_2 are $(t-1)$-balanced. So $(L_x(T_1) \cup xS_1) - (L_x(T_2) \cup xS_2)$ is a $(t+1, t)$ trade, and the result will follow. By Theorem 2.8, $L_x(T)$ has one of two forms.

(1) $L_x(T) = 0(1-2)(3-4)\cdots((2t-1)-2t)$: Any t-subset of $L_x(T)$ that contains 0 occurs in precisely one block in each of $L_x(T_1)$ and $L_x(T_2)$, so will not appear in S_1 or S_2. Any t-subset of $L_x(T)$ not containing 0 appears exactly once in $L_x(T)$, so appears in one of S_1 or S_2. There are exactly 2^t such t-subsets, so $|S_1| = |S_2| = 2^{t-1}$. Any $(t-1)$-subset of these t-subsets must appear in exactly two of them, once in each of S_1 and S_2.

(2) $L_x(T) = (12-34)(5-6)\cdots((2t+1)-(2t+2))$: Since this has exactly t terms, any t-subset of $L_x(T)$ must contain at least one of $\{1,2,3,4\}$. The t-subsets containing 12 or 34 are in a single block of both $L_x(T_1)$ and $L_x(T_2)$, so will not appear in S_1 or S_2. Any t-subset of $L_x(T)$ containing exactly one of $\{1,2,3,4\}$ appears exactly once in $L_x(T)$, so appears in one of S_1 or S_2. There are exactly $4\cdot 2^{t-1}$ such t-subsets, so $|S_1| = |S_2| = 2^t$. Any $(t-1)$-subset, containing one of $\{1,2,3,4\}$, of these t-subsets must appear in exactly two of them, once in each of S_1 and S_2. Any $(t-1)$-subset, containing none of $\{1,2,3,4\}$, of these t-subsets must appear in exactly four of them, twice in each of S_1 and S_2. □

The proof of our result is inductive, with the following result providing our base case.

Lemma 3.24: $13 \notin \mathcal{S}(4,3)$.

Proof: Suppose that $T = T_1 - T_2$ is a $(4,3)$ trade of volume thirteen, and consider any $x \in F(T)$. Note that $r_x \geq 4$, by Lemma 3.14, and that $r_x < 13$, since, if x was in all blocks of T_1 and T_2, then $T_1 \cap T_2 \neq \emptyset$ since T is 3-balanced. Now $R_x(T)$ is a $(3,2)$ trade and $L_x(T)$ is a $(4,2)$ trade. Since $\mathcal{S}(3,2) = \mathcal{S}(4,2) = \{0,4,6,7,8,\ldots\}$, then $r_x = 4$, 6, 7 or 9. By Lemma 3.23 and the final remark in Definition 3.22, $r_x \neq 9$, since $13-8, 13-12 \notin \mathcal{S}(3)$.

We now show that $r_x \neq 4$. Suppose that $r_x = 4$, and that $x \neq y \in F(T)$. Now suppose that x and y do not occur together in a block of T. Since $R_x(T)$ and $R_y(T)$ are disjoint $(3,2)$ trades, the blocks containing x or y are disjoint $(4,2)$ trades. So the remaining blocks of T (that is, $L_y(L_x(T))$) form a $(4,2)$ trade, with volume five, three or two, which is impossible. So any multiplicity four element occurs with every other element of $F(T)$ in some block. Now consider $R_x(T)$. This is a $(3,2)$ trade of volume four, which has a unique structure, with foundation six (Theorem 2.8). But we have just shown that x must occur with every other element of $F(T)$, so $f(T) = 6+1 = 7$, which contradicts $f(T) \geq k+t+1 = 8$.

So $r_x = 6$ or 7. Suppose there are n_6 (resp. n_7) elements of $F(T)$ of multiplicity six (resp. seven). Then $6n_6 + 7n_7 = 13 \times 4 = 52$ has unique solution $n_6 = n_7 = 4$. Thus $f(T) = 8 = 2t + 2$, contradicting Theorem 3.17, as the volume is odd. $\square$

Theorem 3.25: *For all* $t \geq 3$, $s_1 + 1 = 2^t + 2^{t-1} + 1 \notin \mathcal{S}(t+1, t)$.

Proof: We will induct on t, using Lemma 3.24 for the case $t = 3$. So assume that T is a $(t+1, t)$ trade of volume $m = 2^t + 2^{t-1} + 1$ for some $t > 3$, and consider any $x \in F(T)$. Note that $r_x \geq 2^{t-1}$ and $r_x < m$, as in the proof of Lemma 3.24. Now $R_x(T)$ is a $(t, t-1)$ trade and $L_x(T)$ is a $(t+1, t-1)$ trade, so $r_x \in \mathcal{S}(t, t-1)$ and $m - r_x \in \mathcal{S}(t+1, t-1)$. Consequently, $r_x = 2^{t-1}$ or $2^{t-1} + 2^{t-2}$, with $r_x = 2^{t-1} + 2^{t-2} + 1$ disallowed by the induction hypothesis and $r_x = 2^t + 1$ disallowed by Lemma 3.23.

Suppose that there are a (resp. b) elements in $F(T)$ of multiplicity 2^{t-1} (resp. $2^{t-1} + 2^{t-2}$). Thus

$$a2^{t-1} + b(2^{t-1} + 2^{t-2}) = km = (t+1)(2^t + 2^{t-1} + 1).$$

This implies that $t + 1 \equiv 0 \pmod{2^{t-2}}$. That is, $2^{t-2} \mid (t+1)$, which is not possible, since $t \geq 4$. $\square$

3.5 Solutions of $\mathcal{S}[v, k, 1]$ and $\mathcal{S}[v, 3, 2]$

In this section we prove some results on the spectra of trades with particular parameters. We start with the case $t = 1$, and show that the necessary conditions relating v and m quoted so far are sufficient for existence. The following analogue of Lemma 3.7 facilitates our proof of this.

Lemma 3.26: *If* $m \in \mathcal{S}[v, k, 1]$ *then, for all* $1 \leq i \leq m$, $m \in \mathcal{S}[v + i, k + 1, 1]$.

Proof: Let $T = T_1 - T_2$ be a $[v, k, 1]$ trade of volume m and let $e_1, \ldots, e_i$ be i elements not in $F(T)$. Select i blocks of T_1 and i blocks of T_2, and adjoin a single new element to each block, using each of $e_1, \ldots, e_i$ exactly once for the blocks from T_1 and for the blocks from T_2. Adjoining the element e_i to each of the remaining $m - i$ blocks of T_1 and of T_2 gives a $[v + i, k + 1, 1]$ trade of volume m. $\square$

Theorem 3.27:

$$\mathcal{S}[v,k,1] = \begin{cases} \{m : m \geq 2,\ m \equiv 0 \pmod 2\} & \textit{if } k = 2,\ v = 4; \quad \text{(a)} \\ \{m : m \geq 2\} & \textit{if } k \geq 3,\ k+2 \leq v \leq 2k; \quad \text{(b)} \\ \{m : mk \geq v\} & \textit{if } k \geq 2,\ v > 2k; \quad \text{(c)} \\ \emptyset & \textit{otherwise.} \quad \text{(d)} \end{cases}$$

Proof: First, recall that $k \geq t+1 = 2$, $v \geq k+t+1 = k+2$, $m \geq 2^t = 2$ and $v2^{t-1} = v \leq mk$ are necessary. These, with Theorem 3.17, prove (d) and the necessity of the conditions. It remains to show that these conditions are sufficient.

For (a), the sufficiency follows from Theorem 3.16.

For (b), use Theorem 3.16 for the even volumes. For the odd volumes we will construct trades of volume three and then invoke Lemma 3.10 to repeatedly add a suitable trade of volume two. For the $[5,3,1]$ and $[6,3,1]$ trades of volume three, take

$$\begin{aligned} T^5 &= +123+245+345-145-234-235, \\ T^6 &= +123+456+456-145-246-356. \end{aligned}$$

Lemma 3.26 now yields $[v,k,1]$ trades of volume three for all $k \geq 3$ and $k+2 \leq v \leq 3k-3$, which includes the required range for v. For the trades of volume two we use $[k+2,k,1]$ trades, with the $[5,3,1]$ trade being $T^2 = +124+135-125-134$ and the others obtained by using Lemma 3.26. If, whenever we use Lemma 3.26, we always use the smallest possible natural numbers as our new elements, then it is easy to check that we can repeatedly apply Lemma 3.10 to construct the desired trades.

For (c), put $m = \lceil v/k \rceil$. We will consider the cases of volumes $m+2r$ and $m+1+2r$ $(r \geq 0)$ separately.

For the volumes $m+2r$ we will use Lemma 3.10 with the $[k+2,k,1]$ trade of volume two given by $(1-3)(2-4)56\cdots(k+2)$ as our addendum. For the $[v,k,1]$ trade $T_1 - T_2$ of volume m we define two $m \times k$ arrays A and B, the rows of which will be the blocks of T_1 and T_2. The positions of A are filled in row-major order by the elements 1 to v, and the last $mk-v$ positions of the last row of A are filled with the elements 1 to $mk-v$. Array B is formed from A by cycling the first column down by one position: that is, $b_{i,1} = a_{i-1,1}$, with $b_{1,1} = a_{m,1}$.

As an example, when $k=3$ and $v=8$ then $m=3$ and we obtain $T_1 = \{123, 456, 781\}$ and $T_2 = \{723, 156, 481\}$, and the trade of volume two is $+125+345-145-235$. It is straightforward to see that there is never any conflict between the trades of volumes two and m, so Lemma 3.10 now gives all the required volumes.

Using Lemma 3.11, we now construct trades with volumes of the form $m+1+2r$ by adding a basic $[k+2,k,1]$ trade to a trade of volume $m+2r$. This gives all the required trades, except those having $v = 2k+1,\ 2k+2$. However, for these values of v, $m = 2r+4 = (r+2)2$, and these trades exist by Theorem 3.16. $\square$

We now fully determine $\mathcal{S}[v,3,2]$. To do this, we need the following result on Steiner $[v,3,2]$ trades from [11].

Lemma 3.28: ([11]) *For all $n > 0$:*
(1) *A Steiner $[6n,3,2]$ trade of volume $4n+1$ does not exist;*
(2) *A Steiner $[6n+1,3,2]$ trade of volume $4n+1$ does not exist;*
(3) *A Steiner $[6n+3,3,2]$ trade of volume $4n+2$ does not exist;*
(4) *A Steiner $[6n+4,3,2]$ trade of volume $4n+3$ does not exist.* $\square$

To use this result, we will need the following definition and lemma.

Definition 3.29: *Suppose that T is a $[v,3,2]$ trade of volume m. Then $3m-2v$ is called the* ***excess*** *of T.*

Since each element in the foundation of a 2-trade must occur in at least two blocks, the excess can be thought of as the number of places in the trade 'in excess' of those required to use each element twice.

Lemma 3.30: *Suppose that $T = T_1 - T_2$ is a $[v,3,2]$ trade. If T has an excess of at most three, then T is Steiner.*

Proof: We will assume that T is non-Steiner and prove that the excess is at least four. We can suppose that the pair 12 occurs in two blocks of T_1. Recall that if $x \in F(T)$ then $r_x \geq 2$, and note that if $B \in T_1$ then the three 2-subsets of B must occur in separate blocks of T_2, else $T_1 \cap T_2 \neq \emptyset$. There are two cases to consider: without loss of generality, $\{123, 123\} \subseteq T_1$ or $\{123, 124\} \subseteq T_1$.

The first of these is easily dealt with by noting that all six of the pairs 12, 13, 23, 12, 13 and 23 must occur in separate blocks of T_2, giving an excess of at least six.

For the second case, consider the possible arrangements of the six pairs 12, 13, 23, 12, 14 and 24 in T_2. If all six of these are in separate blocks of T_2, then the excess is at least four. If not, then T_2 must contain at least one of the blocks 134 or 234. If T_2 contains both these blocks then, to balance pairs, the pair 34 must occur in two blocks of T_1. So suppose that $\{34a, 34b\} \subseteq T_1$. Then the newly introduced pairs $3a$, $4a$, $3b$ and $4b$ must also occur in blocks of T_2. So both 3 and 4 must occur at least once more in T_2, giving an excess of at least four.

The only subcase remaining is when T_2 contains exactly one of 134 or 234, say 134; note that the pair 34 must now occur in T_1. At this stage, the excess of T_2 is three. Let the pairs 12, 12, 23 and 24 in T_2 occur with a, b, c and d respectively. Note that if $\{1,2,3,4\} \cap \{a,b,c,d\} \neq \emptyset$ then the excess in T_2 is at least four, and the result follows. So suppose that $\{1,2,3,4\} \cap \{a,b,c,d\} = \emptyset$. Now T is also 1-balanced so 1 (resp. 2) must occur at least once (resp. twice) more in T_1. If 1 (resp. 2) occurs more than once (resp. twice) in T_1 then the excess is at least four, and the result follows. So assume that 1 (resp. 2) occurs exactly once (resp. twice) more in T_1 and that 3 and 4 do not occur again in T_1.

Now the newly introduced pairs $a1$, $a2$, $b1$, $b2$, $c2$, $c3$, $d2$ and $d4$ in T_2 must occur in T_1 also. This forces $c = d$ and $c34 \in T_1$, otherwise the excess of T_1 is at least four. So the other sets introduced must be $2ac$, $2bc$ and $1ab$. Now the excess is at least four and the result follows. $\square$

We are now in a position to prove our result.

THEOREM 3.31: *There is a* $[v,3,2]$ *trade of volume* m *if and only if*:

$$\begin{array}{lll} 6 \leq v \leq 6n & \textit{for } m = 4n \textit{ and } n > 0; & \text{(a)} \\ 6 < v \leq 6n-1 & \textit{for } m = 4n+1 \textit{ and } n > 1; & \text{(b)} \\ 6 \leq v \leq 6n+2 & \textit{for } m = 4n+2 \textit{ and } n > 0; & \text{(c)} \\ 6 < v \leq 6n+3 & \textit{for } m = 4n+3 \textit{ and } n > 0. & \text{(d)} \end{array}$$

PROOF: The necessity of the conditions follows from Lemmas 3.30 and 3.28, Lemma 3.14, Theorem 3.17, and $v \geq k+t+1$ and $m(T) \geq 4$. Sufficiency for (a) is immediate, from Theorem 3.16. For the minimal n of (b), (c) and (d), we exhibit example trades for each possible v.

(b) For the volume nine $[v,3,2]$ trades of foundations 7, 8, 9, 10 and 11 use

$$\begin{aligned} &+124+125+136+167+236+246+257+345+456 \\ &\qquad -123-126-146-157-245-245-267-346-356, \\ &+157+168+235+246+278+347+368+458+567 \\ &\qquad -156-178-234-257-268-358-367-457-468, \\ &+147+156+238+246+257+357+369+459+468 \\ &\qquad -146-157-237-248-256-359-368-457-469, \\ &+134+157+168+235+246+279+28a+378+39a \\ &\qquad -135-146-178-234-257-268-29a-379-38a, \\ &+145+168+179+246+257+28a+29b+389+3ab \\ &\qquad -146-157-189-245-268-279-2ab-38a-39b. \end{aligned}$$

(c) For the volume six $[v, 3, 2]$ trades of foundations 6, 7 and 8 use

$$+125 + 134 + 156 + 236 + 246 + 345 - 126 - 135 - 145 - 234 - 256 - 346,$$
$$+125 + 345 + 136 + 246 + 147 + 237 - 126 - 346 - 137 - 247 - 145 - 235,$$
$$+127 + 347 + 567 + 148 + 258 + 368 - 128 - 348 - 568 - 147 - 257 - 367.$$

(d) For the volume seven $[v, 3, 2]$ trades of foundations 7, 8 and 9 use

$$+134 + 157 + 235 + 237 + 246 + 356 + 457 - 135 - 147 - 234 - 236 - 257 - 357 - 456,$$
$$+134 + 156 + 235 + 246 + 278 + 358 + 367 - 135 - 146 - 234 - 258 - 267 - 356 - 378,$$
$$+156 + 179 + 247 + 258 + 269 + 348 + 357 - 157 - 169 - 248 - 256 - 279 - 347 - 358.$$

The trades in (b), (c) and (d) can now be used as in Lemma 3.10 to generate all larger m and v. In all three cases, the $[6, 3, 2]$ basic trade $(1 - 4)(2 - 3)(5 - 6)$ can be used for T^b. □

3.6 Spectrum investigations

Unlike the (v, k, t) trades, the $[v, k, t]$ trades do not form a $\mathbb{Z}$-module. Nonetheless, all $[v, k, t]$ trades can be generated as sums of (v, k, t) basic trades. One basis for the $\mathbb{Z}$-module associated with (v, k, t) trades is the semitriangular basis of smallest trades described in Section 2.2. The construction of this basis has been implemented as a computer programme, and used to search for (k, t) and $[v, k, t]$ trades.

These searches were conducted by first nominating values of v, k and t, and generating the basis. Then, for some fixed $i > 1$, for each combination of i basic trades from the basis (with repetitions allowed or not, as desired), all possible sums/differences of the vectors representing the basis elements were formed. The resulting vectors represent (v, k, t) trades, and are easily processed to extract volume, foundation and structural information. This technique proved capable of yielding useful results concerning the existence of trades up to $t = 8$.

Theoretically, our technique can also demonstrate non-existence, since if a trade of a particular volume exists it can be formed by adding some finite number of basic trades. In practice, however, this exhaustive search is not feasible. Suppose that we wish to investigate t-trades of a particular volume m, with $m = j2^t - x$ for some $j > 1$ and $0 \leq x < 2^t$. Obviously, we must add at least j basis elements when attempting to construct such a trade. If a comprehensive search using a range of values for v, k and $i \geq j$ fails to find any trade of volume m, we regard this as strong evidence that $m \notin \mathcal{S}(t)$.

Our results support Conjecture 2.15(2), but suggest that there are also disallowed volumes in the range above 2^{t+1}. To discuss these results, we start with a generalisation of the s_i.

Definition 3.32: *Fix* $t > 0$ *and let* $0 \leq l \leq t$ *and* $0 \leq i \leq t - l$. *Define*

$$s_i^l = 3 \cdot 2^t - 2^{t-l} - 2^{t-l-i}.$$

The s_i^l are those values that can be obtained from $3 \cdot 2^t$ by subtracting two powers of 2. The larger of these powers is controlled by l, which is called the **level**. Note that when $l = 0$ then $s_i^l = s_i$. The definition of s_i^l leads directly to a proof of the following analogue of Conjecture 2.15(1).

Lemma 3.33: *For all* $0 \leq l \leq t$, $0 \leq i \leq t - l$ *and* $k \geq t + 1$ *there exists a* (k, t) *trade of volume* s_i^l.

Proof: We prove the result for $k = t + 1$, and rely on Lemma 3.7 for all greater k. Given our proof of Conjecture 2.15(1), we need only prove volumes of the form $3 \cdot 2^t - 2^a - 2^b$, with $0 \leq b \leq a \leq t - 1$. Now let

$$\begin{aligned}
T &= \quad (1-2)(3-4)\cdots((2t{+}1)-(2t{+}2)), \\
T^a &= -(1-2)\cdots((2a{-}1)-(2a))((2a{+}1)-(\underline{2a{+}2}))\cdots((2t{+}1)-(\underline{2t{+}2})), \\
T^b &= -(1-2)\cdots((2b{-}1)-(2b))((\underline{2b{+}1})-(2b{+}2))\cdots((\underline{2t{+}1})-(2t{+}2)).
\end{aligned}$$

The trades T and T^a cancel in the 2^a blocks containing $(2a{+}1)\cdots(2t{+}1)$, and T and T^b cancel in the 2^b blocks containing $(2b{+}2)\cdots(2t{+}2)$. As T^a and T^b have no blocks in common, and the two sets of cancelling blocks are disjoint, the result follows. □

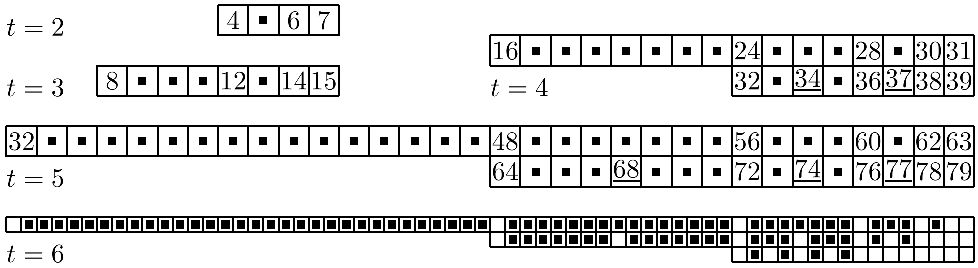
To characterise the pattern of missing volumes observed, we build on selected s_i^l. With each of these we associate a set of 'disallowed' volumes D_i^l. We introduce the parameter $\bar{t} = \lfloor t/2 \rfloor$, as the pattern changes for every increase of t by two.

Definition 3.34: *Fix* $t > 0$ *and, for all* $0 \leq l \leq \bar{t} - 1$ *and* $0 \leq i \leq t - 2l - 2$, *let*

$$D_i^l = \{s_i^l + a2^{t-2l-i-1} + b : 0 \leq a \leq 2^l - 1,\ 1 \leq b \leq 2^{t-2l-i-1} - 1\}.$$

For all other l *and* i, *let* $D_i^l = \emptyset$.

FIGURE 3.1: The sets F_t and $\overline{F_t}$



Note that the s_i^l and the sets D_i^l are disjoint, with the elements in D_i^l falling between s_i^l and s_{i+1}^l. In the pattern of the s_i^l, successive s_i^l on the same level are separated by decreasing powers of two, while moving between levels repeats the last half of the previous pattern. In the pattern of the sets D_i^l, when moving from one level to the next the last half of the previous pattern is repeated, with the difference that each 'range' is split into two by removing the middle value.

DEFINITION 3.35: *For* $t > 0$, *let*

$$\begin{aligned} \overline{F_t} &= \bigcup_{l,i} D_i^l, \\ F_t &= \{s : 2^t \leq s < 3{\cdot}2^t - 2^{t+1-\bar{t}},\ s \notin \overline{F_t}\}. \end{aligned}$$

It is the sets F_t and $\overline{F_t}$ that we will use to describe the observed spectra. They are defined to include only those values up to the highest level that contains non-empty D_i^l. Figure 3.1 illustrates these sets for $2 \leq t \leq 6$. The values are arranged in levels, to highlight how the pattern for $\overline{F_t}$ at one level is obtained by 'halving' the pattern at the previous level. The numbered boxes represent the values of F_t (except for $t = 6$, where space considerations preclude this), while the boxes containing a filled square represent values of $\overline{F_t}$. Level zero is partitioned by the s_i^0 and D_i^0, while in higher levels there are values in F_t that are not values of any s_i^l (for $t = 4$, 5, these values are underlined). Note that some s_i^l are greater than $\max(F_t)$.

In an extensive series of searches using values of $t \leq 8$, no trade with volume in $\overline{F_t}$ was ever constructed. For $1 \leq t \leq 6$ trades of all volumes in F_t were constructed; in fact, trades of all volumes in $\{s : s \geq 2^t\} \setminus \overline{F_t}$ were found. For $t = 7$ all volumes in F_t were constructed, but isolated volumes greater than $\max(F_t)$ remained unconstructed. For $t = 8$ six volumes in F_t remained unconstructed, and isolated volumes greater than $\max(F_t)$ remained unconstructed. Given the large dimensions in these latter two cases, the searches were far from being comprehensive.

For $\mathcal{S}(1)$, $\mathcal{S}(2)$ and $\mathcal{S}(3)$ the non-existent volumes greater than 2^t are precisely those less than 2^{t+1} that are not s_i (except that volume thirteen is undetermined for $\mathcal{S}(k,3)$ when $k > 4$). So our results are intriguing, and suggest that there are disallowed volumes greater than 2^{t+1} when $t > 3$.

We cannot, as yet, prove non-existence for any of the $\overline{F_t}$ that are more than 2^{t+1}. However, we can prove existence for all other volumes when $t = 4$. These techniques can be used for higher values of t, albeit requiring increasing intricacy.

Lemma 3.36: *For all $k > 4$, $\{s : s \geq 16,\ s \notin \overline{F_4}\} \subseteq \mathcal{S}(k,4)$.*

Proof: We prove the result for $k = 5$, and rely on Lemma 3.7 for all greater k. Applying Lemma 3.12 to $\mathcal{S}(4,3) = \{0, 8, 12, 14, 15, 16, \ldots\}$ yields $\{16, 24, 28, 30, 32, \ldots\} \subseteq \mathcal{S}(5,4)$. Since the basic $(5,4)$ trade is simple, Lemma 3.11 now gives $\{31, 39, 43, 45, 47, \ldots\} \subseteq \mathcal{S}(5,4)$. This construction provides all volumes in $\{s : s \geq 16,\ s \notin \overline{F_4}\}$ except for 37 and 41.

Consider the two sets of three basic trades

$$\begin{array}{r}(0-1)(2-3)(4-5)(6-7)(8-9)\\ -(\underline{0}-1)(\underline{2}-3)(4-5)(6-7)(8-9)\\ (\underline{0}-0)(2-3)(\underline{\underline{4}}-5)(\underline{\underline{6}}-7)(\underline{\underline{8}}-9),\end{array}$$

and

$$\begin{array}{r}(0-1)(2-3)(4-5)(6-7)(8-9)\\ -(\underline{0}-1)(\underline{2}-3)(\underline{4}-5)(6-7)(8-9)\\ (\underline{0}-0)(2-3)(\underline{\underline{4}}-5)(\underline{\underline{6}}-7)(\underline{\underline{8}}-9).\end{array}$$

When added, the first set of basic trades yields a trade of volume 37, and the second set yields a trade of volume 41. To see this, simply note that $37 = 48 - 8 - 2 - 1$ and $41 = 48 - 4 - 2 - 1$, and that, for example, the first pair of basic trades cancel in 8 blocks. □

3.7 Conjectures

It is tempting to speculate that $\mathcal{S}(t)$ contains all volumes of at least 2^t except those in $\overline{F_t}$. However, the evidence in support of this is largely empirical, and extends only to $t = 8$; it is possible that for larger values of t a more complicated pattern of disallowed volumes would emerge. Obviously, $\mathcal{S}(t)$ needs to be further investigated. We do, however, offer the following extended version of Conjecture 2.15.

CONJECTURE 3.37:

(1) *For all* $s \in F_t$, *there exists a* (k,t) *trade of volume* s.

(2) *For all* $s \in \overline{F_t}$, *no* (k,t) *trade of volume* s *exists.*

Our theoretical results and empirical data suggest that $\mathcal{S}(k,t)$ does not depend on k, and we therefore make the following conjecture.

CONJECTURE 3.38: *For all* $k > t > 0$, $\mathcal{S}(k,t) = \mathcal{S}(t)$.

Since basic $[v, t+1, t]$ trades must have $v = 2t+2$, then $2^t \notin \mathcal{S}[v, t+1, t]$ when $v > 2t+2$. Since $[2t+2, t+1, t]$ trades must have even volume, then $s_t \notin \mathcal{S}[2t+2, t+1, t]$. So, for all v, the inclusion $\mathcal{S}[v, t+1, t] \subseteq \mathcal{S}(t)$ is proper. For any t and $k > t+1$, is it possible to find a value of v such that $\mathcal{S}[v,k,t] = \mathcal{S}(t)$? We have proved that $\mathcal{S}[5,3,1] = \mathcal{S}(1)$. That $\mathcal{S}[7,4,2] = \mathcal{S}(2)$ is easily shown. Our computer searches indicate that $\mathcal{S}[2t+3, t+2, t] = \mathcal{S}(t)$ for $t = 3$, 4 and 5, if Conjecture 3.37(2) is true. Replacing $2t+3$ by arbitrary v, we make the following conjecture.

CONJECTURE 3.39: *For any* $t > 0$, *there exists a* v *such that* $\mathcal{S}[v, t+2, t] = \mathcal{S}(t)$.

Note that if this conjecture is true, then Lemma 3.7 implies that such a v exists for all $k > t+1$.

Theorem 3.21 leaves undecided the existence of $[v_i, k, t]$ trades of volume s_i, $1 \le i \le t$, for the range $3k - t + i < v_i \le 4k - k/2^{i-1}$. We conjecture that these do not exist.

CONJECTURE 3.40: *If* T *is a* $[v,k,t]$ *trade of volume* s_i, $1 \le i \le t$, *then* $v \le 3k - (t-i)$.

We are unsure of the exact form of the upper bound for v in general. However, our computer searches support the following conjecture.

CONJECTURE 3.41: *If* T *is a* $[v,k,t]$ *trade of volume* $m < j2^t$, *then* $v \le (2j-1)k$.

Note that the best bound obtainable using Lemma 3.14 is $v < 2jk$. Of course, when $m = j2^t$, the possible values for v are given by Theorem 3.16.

3.8 The structure of $[v,k,2]$ trades when $2v = mk$

Recall Lemma 3.14; when $t = 2$, this yields $2v \le mk$. In this section we completely characterise those $[v,k,2]$ trades where $2v = mk$.

DEFINITION 3.42: *A partition of a trade T into subtrades is said to be a* ***decomposition*** *of T. If all the subtrades in a decomposition are basic trades, then the decomposition is said to be a* ***basic decomposition****.*

The trades constructed in Theorem 3.16 all have a basic decomposition. In fact, the trades of Theorem 3.16 include all those $[v, k, 2]$ trades that attain the bound $2v = mk$.

THEOREM 3.43: *Suppose that $T = T_1 - T_2$ is a $[v, k, 2]$ trade of volume m. If $2v = mk$ then T is either a $[2k, k, 2]$ basic trade or T has a unique basic decomposition, with the basic trades in the decomposition being $[2k, k, 2]$ basic trades on disjoint foundations. Thus T is a $[2rk, k, 2]$ trade of volume $4r$, and such trades exist for all $k > 2$ and $r > 0$.*

PROOF: First note that the maximum foundation for a basic trade is $2k$, and that such maximum trades have a unique form and always exist (Theorem 2.8 and Lemma 3.15). So we can assume that $m > 4$. Existence of a $[2rk, k, 2]$ trade of volume $4r$ follows from Theorem 3.16. It remains to prove that T has a unique basic decomposition into $[2k, k, 2]$ trades on disjoint foundations.

By Lemma 3.14, $r_x = 2$ for all $x \in F(T)$. So let A_1 and B_1 be the two blocks in T_1 that contain $x \in F(T)$, and suppose that $|A_1 \cap B_1| = i$. Now if $i = k$ or $i = k - 1$, then balancing pairs containing x forces $T_1 \cap T_2 \neq \emptyset$. So $0 < i < k - 1$, and we can assume that

$$\begin{aligned} A_1 &= xa_1 \cdots a_{i-1}b_1 \cdots b_{k-i}, \\ B_1 &= xa_1 \cdots a_{i-1}c_1 \cdots c_{k-i}, \end{aligned}$$

where x and the a's, b's and c's are distinct elements. Let A_2 and B_2 be the two blocks in T_2 that contain x. Then, without loss of generality, we can write

$$\begin{aligned} A_2 &= xa_1 \cdots a_{i-1}b_1 \cdots b_j c_1 \cdots c_{k-i-j}, \\ B_2 &= xa_1 \cdots a_{i-1}b_{j+1} \cdots b_{k-i}c_{k-i-j+1} \cdots c_{k-i}, \end{aligned}$$

for some $0 < j < k - i$.

Now all pairs of the form $b_r b_s$ ($1 \leq r \leq j$, $j+1 \leq s \leq k-i$) and $c_r c_s$ ($1 \leq r \leq k-i-j$, $k - i - j + 1 \leq s \leq k - i$) appear in blocks of T_1 but not T_2. Since each of the b's and c's must occur in exactly one more block of T_2, it must contain the partial blocks $X_2 = b_1 \cdots b_{k-i}$ and $Y_2 = c_1 \cdots c_{k-i}$. Similarly, the partial blocks $X_1 = b_1 \cdots b_j c_1 \cdots c_{k-i-j}$ and $Y_1 = b_{j+1} \cdots b_{k-i}c_{k-i-j+1} \cdots c_{k-i}$ must be in T_1.

If $X_1 \cup Y_1$ is in a block of T_1 then to balance pairs such as b_1c_{k-i} we must have $X_2 \cup Y_2$ in a block of T_2. But now pairs such as b_1b_{k-i} are not balanced. So X_1 and Y_1 (and similarly, X_2 and Y_2) are in separate blocks of T_1 (resp. T_2), say C_1 and D_1 (resp. C_2 and D_2).

Now consider any $d \in C_1 \setminus X_1$. To balance pairs such as db_1 and dc_1 (say), d must be in both X_2 and Y_2. Now d occurs with all of the b's and c's in T_2, so $d \in D_1$. Thus all of C_1, D_1, C_2 and D_2 contain the partial block $d_1 \cdots d_i$ for some new set of distinct d's.

Now $R = +A_1 + B_1 + C_1 + D_1 - A_2 - B_2 - C_2 - D_2$ is a $[2k, k, 2]$ trade of volume four; that is, a basic trade of maximum foundation. But now $T - R$ is a $[v - 2k, k, 2]$ trade of volume $m - 4$, and $2(v - 2k) = (m - 4)k$. So $T - R$ is either a $[2k, k, 2]$ basic trade or we can recursively decompose it into a $[2k, k, 2]$ basic trade and a $[v - 4k, k, 2]$ trade of volume $m - 8$. Since m is finite, the result now follows. $\square$

3.9 Trades of volumes s_1

Having established the existence of (k, t) trades of volume m, and the values of v for which such trades exist, we might now ask, "how many non-isomorphic trades are there?" One motivation for this is to enable us to enumerate, via some form of exhaustive search, all the trades of a particular volume in a design.

The structure of basic trades is given by Theorem 2.8. When $k = t+1$, basic trades are unique and are Steiner. The number of different basic trades grows rapidly for larger k, as the number of ways of partitioning $k - t - 1$, the maximum size of the trade's tail, increases [75]. When $k > t + 1$ basic trades are always simple and, except in the case $t = 1$, are never Steiner.

The next case to consider is volume $s_1 = 2^t + 2^{t-1}$. We first discuss the case $k = t+1$ for $t = 1$ and $t = 2$. When $t = 1$, it is trivial to verify that there are exactly two non-isomorphic $(2, 1)$ trades of volume three; the simple non-Steiner $[5, 2, 1]$ trade $+12+13+45-14-15-23$, and the Steiner $[6, 2, 1]$ trade $+12+34+56-13-25-46$.

Lemmas 3.18 and 3.20 yield $(3,2)$ trades of volume six with foundations 6, 7 and 8.

$$\begin{aligned}
T^a &= +146+236+245+125+134+635\\
&\quad -136-145-235-124-625-634,\\
T^b &= +146+236+245+13\underline{6}+165+26\underline{6}\\
&\quad -136-145-246-16\underline{6}-23\underline{6}-265,\\
T^c &= +146+236+245+13\underline{6}+1\underline{4}5+2\underline{46}\\
&\quad -136-145-246-1\underline{46}-23\underline{6}-2\underline{4}5.
\end{aligned}$$

All three of these are simple, but only T^c is Steiner. Note that T^b and T^c could also have been obtained from the $(2,1)$ trades via Lemma 3.12.

Another $[7,3,2]$ trade of volume six is

$$\begin{aligned}
T^d &= +145+167+246+257+347+356\\
&\quad -146-157-247-256-345-367.
\end{aligned}$$

This trade is Steiner and, like T^c, is described in Lemma 4.20 of the next chapter. In the terminology of the next chapter, T^a is *regular* and T^d is *linked.* Note that none of these four trades contains a subtrade. However, T_1^d and T_2^d are non-minimal; the halves of the other trades are minimal.

LEMMA 3.44: *If T is a $(3,2)$ trade of volume six, then T is isomorphic to one of the four trades T^a, T^b, T^c, T^d exhibited above.*

PROOF: If T is Steiner, we prove in Lemma 4.20 of the next chapter that it must be isomorphic to T^c or T^d. Thus we need only prove the non-Steiner case here. If T is also non-simple, then we can assume that $\{123, 123\} \in T_1$. To match pairs, and since $T_1 \cap T_2 = \emptyset$, the six pairs 12, 13, 23, 12, 13 and 23 must occur in separate sets of T_2. Since $f(T) \geq k+t+1$, and $r_x \geq 2$ for all $x \in F(T)$, then $f(T) = 6$ and the other elements of $F(T)$, say 4, 5 and 6, must have multiplicity 2. It is now easy to check that, no matter how, say, 4 is placed in T_2, it must occur at least thrice in T_2, a contradiction.

So we can assume that T is simple, but non-Steiner; that is, we assume that $\{123, 124\} \in T_1$. Of the six pairs 12, 13, 23, 12, 14 and 24 that must appear in T_2, the four pairs 12, 13, 23 and 12 must appear in separate sets. The elements that appear with 12 and 12 must be distinct from 4 and from each other (since T is simple); say they are 5 and 6. Note that the pairs 15, 25, 16 and 26 must now occur in T_1. Now the pairs 14 and 24 must appear in T_2; we consider the three cases.

(1): If 4 occurs with 13 and 23 in T_2, then the pair 34 must occur twice in T_1. These must be in two of the remaining 4 sets of T_1, and this forces 156 and 256 to be in T_1. This, in turn, forces the pair 56 to occur in the two remaining sets of T_2. This forces 345 and 346 in T_1 and 356 and 456 in T_2, and T is isomorphic to T^a.

(2): Exactly one of the pairs 14 and 24 appears separately in T_2. Suppose, without loss of generality, that 4 occurs with 13 and 24 occurs separately in T_2. Now the five pairs 15, 25, 16, 26 and 34 must appear in the four undetermined sets of T_1. Thus 34 is in a separate set of T_1. Since 56 can appear in T_2 once only, T_1 contains either 156, 25 and 26 or 256, 15 and 16. In either case, it is easy to check that the three positions remaining in T_1 cannot contain any of 1,. . . ,6. So these positions must all contain a new element, 7 say. In the first case T is isomorphic to T^b. In the second, T is not a trade.

(3): If the pairs 14 and 24 appear separately in T_2, then a total of four positions of T_2, all singletons, remain unfilled. The pairs 15, 25, 16 and 26 must now occur in T_1 in separate sets, else the pair 56 would be present and this pair cannot be in T_2. This partial completion of T is 2-balanced, but not 1-balanced, since, for example, 3 must occur again in T_1. It is easy to check that 3 cannot occur in any of the four unfilled positions of T_1. For example, if $153 \in T_1$, then the pairs 13 and 53 must be added to T_2. This forces $235, 143 \in T_2$, and the newly introduced pair 34 cannot be in T_1. The other positions are similar, and so this case cannot occur. □

For general k, t and i, Lemmas 3.18 and 3.20, along with repeated application of Lemmas 3.7 and 3.12, can be used to generate simple trades of volume s_i with many different foundations and structures – although not all trades can be obtained in this way. The number of different trades grows rapidly; for example, there are at least ten non-isomorphic simple non-Steiner minimal $(4, 2)$ trades of volume six [93, Table 5.12]. To obtain non-simple (k, t) trades of volume s_i for all t and $k > t + 1$ the $[6, 3, 1]$ trade $+123 + 123 + 456 - 124 - 135 - 236$ can be used.

REMARK: No non-simple $(t + 1, t)$ trades of volume $m < 2^{t+1}$ are known.

3.10 Conclusions

The study of $\mathcal{S}[v, k, t]$ and its related spectra is a challenging problem. Although all trades can be generated as sums of basic trades, using this knowledge to determine spectra and foundations of trades in general is not necessarily straightforward. Our technique of manipulating the exact form of the basic trades to be added is powerful, and enables us to construct trades with a nominated volume or foundation. However, it becomes unwieldy as the number of basic trades to be added, and hence the number of potential pair-wise cancellations, increases. It also does not address the problem of proving non-existence, which seems to be very difficult.

Our computerised search for trades was very successful, and was the inspiration for many of the results and conjectures presented in this chapter. Our programme produced general trades efficiently. However, simple and Steiner trades are much ‘harder’ to generate. The particular basis used is only one of the inequivalent bases available [76]. It would be interesting to implement some of these other bases as computer programmes, and assess their effectiveness in generating trades having particular properties.

Chapter 4

Steiner trades

Recall that a trade $T_1 - T_2$ where each t subset appears at most once in T_1, and thus also in T_2, is a Steiner trade.

Notation: For the spectrum of Steiner (k, t) trades, we specialise the notation $\mathcal{S}(k, t)$ to $\mathcal{S}_1(k, t)$. We also define $\overline{\mathcal{S}}_1(k, t) = \{m : m \geq 0,\ m \notin \mathcal{S}_1(k, t)\}$.

It is $\mathcal{S}_1(k, t)$ that we study in this chapter. It is already known that $\mathcal{S}_1(3, 2) = \{0, 4, 6, 7, 8, \ldots\}$, $\mathcal{S}_1(4, 2) = \{0, 6, 8, 9, 10, \ldots\}$ and $\mathcal{S}_1(4, 3) = \{0, 8, 12, 14, 15, 16, \ldots\}$ [7]. We determine $\mathcal{S}_1(5, 2)$ and $\mathcal{S}_1(6, 2)$, and leave only the odd volumes m, $2k+1 < m < 3k-3$, undetermined for $\mathcal{S}_1(k, 2)$, $k \geq 7$. Additionally, the structure of trades of certain volumes is categorised. We also briefly consider the case $t > 2$, both when $k = t+1$ and when $k > t+1$. Our main result, which we prove in parts throughout this chapter, is the following.

Theorem 4.1: (1) *If* $0 < m < 2k-2$ *or* $m = 2k-1$, *then* $m \notin \mathcal{S}_1(k, 2)$;
(2) *If* $m \geq 3k-3$, *or* m *is even and* $2k-2 \leq m \leq 3k-4$, *then* $m \in \mathcal{S}_1(k, 2)$;
(3) *A Steiner* $(k, 2)$ *trade of volume* $2k-2$ *has a unique structure;*
(4) *A Steiner* $(k, 2)$ *trade of volume* $2k$ *has a unique structure precisely when* $k > 4$;
(5) $2k+1 \in \mathcal{S}_1(k, 2)$ *precisely when* $k \in \{3, 4, 7\}$;
(6) *A Steiner* $(k, 2)$ *trade of volume* $2k+1$ *has a unique structure precisely when* $k = 7$.

Appendix C contains an account of the *swap matrix* technique developed to prove some specific cases. Appendix D shows how some of our work can applied to the more general concept of a *G-trade*.

The work reported in Sections 4.1–4.6 and Theorem 4.42 of this chapter, and in Appendices C and D, has been accepted for publication [40, 41].

4.1 Preliminary results

For completeness, we first determine $\mathcal{S}_1(k, 1)$.

THEOREM 4.2: *$\mathcal{S}_1(k,1) = \{0,2,3,4,\ldots\}$. Further, if $T = T_1 - T_2$ is a $(k,1)$ Steiner trade of volume m, then T_1 consists of m disjoint k-subsets.*

PROOF: Clearly, the volume of a (k,t) trade cannot equal one. If T_1 is a Steiner $(k,1)$ trade, then the blocks of T_1 must be disjoint or else a 1-subset is repeated. Let T_1 be any collection of $m \geq 2$ mutually disjoint k-subsets $\{A_1, A_2, \ldots, A_m\}$ and choose $x_i \in A_i$ for $i = 1,2,\ldots,m$. Let $B_i = A_i \setminus \{x_i\}$ for $i = 1,2,\ldots,m$. Then $T_1 = \{x_1B_1, x_2B_2, \ldots, x_mB_m\}$ trades with $T_2 = \{x_mB_1, x_1B_2, x_2B_3, \ldots, x_{m-1}B_m\}$ and $T_1 - T_2$ is a Steiner $(k,1)$ trade of volume m. $\square$

LEMMA 4.3: *Suppose $m_1, m_2 \in \mathcal{S}_1(k,t)$, where m_1 and m_2 may be equal. Then:*
(1) $m_1 + m_2 - 1 \in \mathcal{S}_1(k,t)$;
(2) $m_1 + m_2 \in \mathcal{S}_1(k,t)$. That is, $\mathcal{S}_1(k,t)$ is closed under addition.

PROOF: Let $T^a = T_1^a - T_2^a$ and $T^b = T_1^b - T_2^b$ be Steiner $(k,2)$ trades such that $m(T^a) = m_1$ and $m(T^b) = m_2$.

(1) We first show that $m_1 + m_2 - 1 \in \mathcal{S}_1(k,t)$. Choose a block B of T_1^a. Relabel the elements of any block C of T_2^b so that $C = B$. Now relabel the elements of $F(T^b) \setminus B$ so that $F(T^a) \cap (F(T^b) \setminus B) = \emptyset$. To show that $T^a + T^b$ is Steiner we need only consider t-subsets of B because of the choice of $F(T^b)$. However, since T^a and T^b are Steiner, any t-subset S of B occurs precisely once in each of T_1^a, T_2^a, T_1^b and T_2^b. Since B is common to T_1^a and T_2^b, $B \notin T_1^a - T_2^b$ and so S occurs precisely once in each half of $T^a + T^b$. Finally, as B is the only block common to T^a and T^b, $m(T^a + T^b) = m_1 + m_2 - 1$.

(2) To show that $m_1 + m_2 \in \mathcal{S}_1(k,t)$, relabel the elements of $F(T^b)$ so that $F(T^a) \cap F(T^b) = \emptyset$. Then $T^a + T^b$ is a Steiner (k,t) trade of volume $m_1 + m_2$. $\square$

We illustrate the construction method of Lemma 4.3(1) with the following example.

EXAMPLE 4.4: Two Steiner $(3,2)$ trades of volume six are

$$\begin{aligned} T^a &= T_1^a - T_2^a = +134 + 156 + 178 + 235 + 247 + 268 \\ &\qquad\qquad -135 - 147 - 168 - 234 - 256 - 278, \\ T^b &= T_1^b - T_2^b = +134 + 156 + 235 + 246 + 036 + 045 \\ &\qquad\qquad -135 - 146 - 236 - 245 - 034 - 056. \end{aligned}$$

We show how to construct a Steiner $(3,2)$ trade of volume eleven. Let $B = 134 \in T_1^a$ and $C = 135 \in T_2^b$. We form T^c by first transposing the elements $4,5 \in F(T^b)$ so that $B = C$ and then relabelling the elements of $F(T^b) \setminus B$ to $\{\underline{0}, \underline{2}, \underline{5}, \underline{6}\}$ so that

$F(T^a) \cap \{\underline{0}, \underline{2}, \underline{5}, \underline{6}\} = \emptyset$. Thus,

$$\begin{aligned} T^c \;=\; T_1^c - T_2^c \;=\; & +13\underline{5} + 14\underline{6} + \underline{2}34 + \underline{256} + \underline{0}3\underline{6} + \underline{0}4\underline{5} \\ & -134 - 1\underline{56} - \underline{2}3\underline{6} - \underline{2}4\underline{5} - \underline{0}3\underline{5} - \underline{0}4\underline{6}. \end{aligned}$$

It is simple to check that $T^a + T^c$ is a Steiner $(3, 2)$ trade of volume eleven. $\square$

Lemma 4.3 is a very powerful result since, if a single non-void Steiner (k, t) trade exists, the problem of determining $\mathcal{S}_1(k, t)$ reduces to a finite number of cases.

Theorem 4.5: *Exactly one of the following is true:*
(1) $\mathcal{S}_1(k, t) = \{0\}$*;*
(2) $\overline{\mathcal{S}}_1(k, t)$ *is finite.*

Proof: If (1) is true, then (2) is trivially false. If $\mathcal{S}_1(k, t) \neq \{0\}$, then $m \in \mathcal{S}_1(k, t)$ for some $m > 0$. By Lemma 4.3(1), $2m - 1 \in \mathcal{S}_1(k, t)$. As m and $2m - 1$ are coprime and $\mathcal{S}_1(k, t)$ is closed under addition, $\overline{\mathcal{S}}_1(k, t)$ is finite. $\square$

Corollary 4.6: $\overline{\mathcal{S}}_1(t + 1, t)$ *is finite.*

Proof: In Theorem 2.8 the tail of a basic $(t + 1, t)$ trade is empty and all the S_i, $1 \leq i \leq 2t + 2$, are singletons. So basic $(t + 1, t)$ trades are Steiner. $\square$

4.2 Solely t-balanced families

Definition 4.7: *Let A^1 and A^2 be collections of s-subsets of V. If $A^1 - A^2$ is a Steiner (s, t) trade and A^1 and A^2 contain no common $(t+1)$-subset, then A^1 and A^2 are said to be* ***solely*** *t-balanced. A set $\{A^1, \ldots, A^m\}$ of collections of s-subsets such that A^i and A^j are solely t-balanced for each $i \neq j$ is said to be a* ***solely t-balanced family***.

The following analogue of Lemma 3.12 illustrates how solely t-balanced families can be used to generate Steiner $(t + 1)$-trades.

Lemma 4.8: *Let A^1 and A^2 be collections of $(k - 1)$-subsets such that $\{A^1, A^2\}$ is a solely t-balanced family. Choose distinct $x, y \notin F(A^1)$, and let $T_1 = xA^1 + yA^2$ and $T_2 = xA^2 + yA^1$. Then $T = T_1 - T_2$ is a Steiner $(k, t + 1)$ trade.*

Proof: By Lemma 3.12, T is a $(k, t+1)$ trade; it remains to show that T is Steiner. Let α be a $(t + 1)$-subset of elements in a block of T_1. It suffices to show that α is contained in precisely one k-subset in each of T_1 and T_2. There are two cases to consider.

Case 1: Suppose $\alpha \cap \{x, y\} = \emptyset$. Without loss of generality, suppose α is contained in a block of A^1. As A^1 and A^2 are solely t-balanced, α does not occur in A^2. Hence α is contained exactly once in each of T_1 and T_2.

Case 2: Suppose $\alpha \cap \{x, y\} \neq \emptyset$. In this case, without loss of generality, suppose that $x \in \alpha$ and that the t-subset $\alpha \setminus \{x\}$ is contained in a block of A^1. But $\alpha \setminus \{x\}$ is contained precisely once in each of A^1 and A^2, as $A^1 - A^2$ is a Steiner $(k-1, t)$ trade. Thus α is contained precisely once in each of T_1 and T_2. □

Example 4.9: Let $A^1 = \{12, 34\}$, $A^2 = \{13, 24\}$ and x, y be distinct elements not equal to $1, 2, 3$ or 4. Then A^1 and A^2 are solely 1-balanced, and

$$\begin{aligned} T &= xA^1 + yA^2 - xA^2 - yA^1 \\ &= +x12 + x34 + y13 + y24 - x13 - x24 - y12 - y34 \end{aligned}$$

is a Steiner $(3, 2)$ trade; in fact, the Pasch trade. □

Our primary concern in this chapter is Steiner $(k, 2)$ trades, and our main tool for constructing these is solely 1-balanced families. These families are straightforward to construct, and we now give the particular construction we use.

Notation: Let $A(k)$ equal the $(k-1) \times (k-1)$ array with entries

$$a_{ij} = (i-1)(k-1) + j, \qquad \text{for } i, j = 1, 2, \ldots, (k-1).$$

Write R, C and F for the collections of elements of each of the rows, columns and forward diagonals of $A(k)$ respectively, suppressing k, since it will be fixed. Define $\underline{A}(k, r)$, $1 \leq r \leq k-1$, to be the $(k-1) \times (k-1)$ array with each of the elements a_{ij} of the first r rows of $A(k)$ replaced by $\underline{a}_{ij}$, where

$$\underline{a}_{ij} \neq a_{lm} \text{ for } 1 \leq i, l \leq r,\ 1 \leq j, m \leq k-1.$$

$\underline{R}_r$ and $\underline{C}_r$ are the sets of elements of each of the rows and columns of $\underline{A}(k, r)$ respectively. It is easy to see that $\{\underline{R}_r, \underline{C}_r\}$ and $\{R, C, F\}$ are solely 1-balanced families.

Example 4.10: Let $k = 4$. Then,

$$\begin{aligned} R &= \{123, 456, 789\}, & C &= \{147, 258, 369\}, \\ \underline{R}_1 &= \{\underline{123}, 456, 789\}, & \underline{C}_1 &= \{\underline{1}47, \underline{2}58, \underline{3}69\}, \\ F &= \{159, 267, 348\}. \end{aligned}$$

□

4.3 The spectrum of Steiner $(k, 2)$ trades

In this section, we determine $\mathcal{S}_1(k, 2)$ for all k, except for the odd volumes in the range $2k + 1$ to $3k - 4$. To prove our results, we will need some technical lemmas regarding the multiplicity of a set of elements in the blocks of a trade. We start by extending our definition of multiplicity.

DEFINITION 4.11: *For an s-subset S and trade $T = T_1 - T_2$, let $r_S(T_1)$ equal the number of blocks in T_1 containing S. We say that S has* ***multiplicity*** *$r_S(T_1)$ in T_1. If $s \leq t$, then $r_S(T_1) = r_S(T_2)$, and so we define $r_S(T) = r_S(T_1)$ in this case. If $S = \{x\}$, we write r_x for $r_x(T) = r_{\{x\}}(T_1)$, as before. Also define $r(T) = \min\{r_x : x \in F(T)\}$.*

LEMMA 4.12: *If S is an s-subset, where $1 \leq s < t$, and T is a (k, t) trade, then*

$$r_S(T) \neq 1,\ m(T) - 1.$$

PROOF: See, for example, the proof of Lemma 3 in [67]. □

COROLLARY 4.13: *If T is a (k, t) trade and $t > 1$, then $r(T) \geq 2$.* □

LEMMA 4.14: *If T is a Steiner $(k, 2)$ trade and $x \in F(T)$, then*

$$2 \leq r_x \leq \frac{k\, m(T)}{(2k - 1)}.$$

PROOF: The $r_x(k - 1)$ elements occurring with x in T_1 must all be distinct, since the trade is Steiner. Since $r(T) \geq 2$ by Corollary 4.13, each of these elements must appear at least once more. Thus, $k\, m(T) \geq r_x + 2r_x(k - 1)$ and so $r_x \leq k\, m(T)/(2k - 1)$. □

LEMMA 4.15: *If $T = T_1 - T_2$ is a Steiner $(k, 2)$ trade with $r(T) > 2$, then $m(T) \geq 2k + 1$.*

PROOF: If $a_1 a_2 \ldots a_k$ is a block of T_1, then the elements $a_1, a_2, \ldots, a_k$ each occur at least two more times in the blocks of T_1 since $r(T) > 2$. As no pair of these elements can occur in more than one block of T_1, there are at least $2k + 1$ blocks in T_1. □

LEMMA 4.16: *Suppose $T = T_1 - T_2$ is a Steiner $(k, 2)$ trade, with $r_\alpha = 2$ for some $\alpha \in F(T_1)$ and either (1) $m(T) < 4k - 10$; or (2) $m(T) \leq 2k - 1$. If B_1 and B_2 are the two blocks of T_1 containing α, then there exist distinct elements $x \in B_1$ and $y \in B_2$ such that at least $k - 1$ blocks of T_1 contain x but not y, and at least $k - 1$ blocks of T_1 contain y but not x.*

Proof: For $\alpha \in F(T)$ with $r_\alpha = 2$, without loss of generality, let $B_1 = \alpha a_1 \dots a_{k-1}$ and $B_2 = \alpha b_1 \dots b_{k-1}$ represent the two distinct blocks in T_1 containing α. Let C_1 and C_2 represent the two blocks in T_2 also containing α. As T_1 and T_2 contain precisely the same pairs involving α, each of $a_1, \dots, a_{k-1}, b_1, \dots, b_{k-1}$ is contained in exactly one of C_1, C_2. Let $j_a = |(B_1 \cap C_1) \setminus \{\alpha\}|$, so $1 \leq j_a \leq k-2$. Relabel C_1, C_2 and the elements of B_1 and B_2 as necessary so that, without loss of generality,

$$C_1 = \alpha a_1 \dots a_{j_a} b_1 \dots b_{j_b}, \qquad C_2 = \alpha a_{j_a+1} \dots a_{k-1} b_{j_b+1} \dots b_{k-1},$$

where $j_a + j_b = k-1$ and $1 \leq j_a \leq j_b$. Note that, subject to these constraints, the minimum of the product $j_a j_b$ occurs when $(j_a, j_b) = (1, k-2)$.

Let $i_1 \in \{1, \dots, j_a\}$, $i_2 \in \{1, \dots j_b\}$, $i_3 \in \{j_a+1, \dots, k-1\}$, $i_4 \in \{j_b+1, \dots, k-1\}$. The pairs $\{a_{i_1}, b_{i_2}\}$ and $\{a_{i_3}, b_{i_4}\}$ occur in T_2 and thus must also occur in T_1. Moreover, each of these pairs must occur in a separate block of T_1 not containing α as T is a Steiner trade. This implies $m(T) \geq 2 + j_a j_b + (k-1-j_a)(k-1-j_b) = 2 + 2 j_a j_b$.

If $(j_a, j_b) \neq (1, k-2)$, then $j_a j_b \geq 2(k-3)$ which implies that $m(T) \geq 2 + 4(k-3) = 4k-10$. So if (1) is true, then $(j_a, j_b) = (1, k-2)$. If (2) is true (and (1) is not), then $4k - 10 \leq m(T) \leq 2k-1$ which implies $k \leq 4$. For $k = 3, 4$, j_a and j_b are uniquely determined as 1 and $k-2$ respectively. Thus $(j_a, j_b) = (1, k-2)$ in all cases and the element a_1 occurs in blocks with $b_1, b_2, \dots, b_{k-2}$. As $B_2 \in T_1$ and T is Steiner, each of these $k-2$ blocks in T_1 is distinct and cannot contain b_{k-1}. However block B_1 also contains a_1 and does not contain b_{k-1}, and thus there are at least $k-1$ blocks in T_1 containing a_1 and not b_{k-1}. By symmetry, there are at least $k-1$ blocks containing b_{k-1} and not a_1 in T_1. Letting $x = a_1$ and $y = b_{k-1}$ yields the result. □

Lemma 4.17: *Suppose $T = T_1 - T_2$ is a Steiner $(k, 2)$ trade and there exist distinct elements $x, y \in F(T)$ such that either*
(1) $k = 3$, $r_x + r_y = m(T)$ and $r_{\{x,y\}} = 0$; or
(2) $k > 3$, $r_x + r_y \geq m(T)$.
Then $T_1 = xA^1 + yA^2$ and $T_2 = xA^2 + yA^1$ where A^1 and A^2 are solely 1-balanced and $x, y \notin F(A^1)$. Thus $r_x = r_y = m(T)/2$.

Proof: The proof of (1) when $k = 3$ is simple and thus omitted. So, assume that $k > 3$. For (2), we first show that $r_{\{x,y\}} = 0$. Suppose not; then $r_{\{x,y\}} = 1$ as T is Steiner, and B, say, is the unique block in T_1 containing both x and y. There is at most one block in T_1 which contains neither x nor y. Thus any two elements distinct from x and y which occur in B cannot both occur again in T_1 without repeating a pair. Hence $r_{\{x,y\}} = 0$.

It is now obvious that $r_x + r_y = m(T)$ and the blocks of T_1 can be written as $\{xA^1, yA^2\}$ for some collections A^1, A^2 of disjoint $(k-1)$-subsets. However, as each element in A^1 must occur at least twice in the blocks of T_1 by Lemma 4.12, it follows that $F(A^1) = F(A^2)$ and $\{A^1, A^2\}$ is a solely 1-balanced family.

It is easy but tedious to show that any block (that is, $(k-1)$-subset) S of A^1 must also occur in T_2. It then follows that $T_2 = xA^2 + yA^1$ as claimed. □

Lemma 4.16 allows us to prove the following two non-existence results for the volumes of Steiner $(k, 2)$ trades. Additionally, using Lemma 4.17, the structure of Steiner $(k, 2)$ trades with certain volumes can be determined.

THEOREM 4.18: *If $T = T_1 - T_2$ is a Steiner $(k, 2)$ trade, then $m(T) \geq 2k - 2$. If $m(T) = 2k - 2$, then $T_1 = xA^1 + yA^2$ and $T_2 = xA^2 + yA^1$ where A^1 and A^2 are solely 1-balanced and $x, y \notin F(A^1)$.*

PROOF: If $m(T) < 2k - 2$, then by Lemma 4.15 $r(T) = 2$. By Lemma 4.16, there exist at least $k-1$ blocks which contain x but not y and at least $k-1$ blocks which contain y but not x. Thus $m(T) \geq 2(k-1)$, a contradiction. When $m(T) = 2(k-1)$, then the structure of T follows from Lemmas 4.15, 4.16 and 4.17. □

THEOREM 4.19: *If $T = T_1 - T_2$ is a Steiner $(k, 2)$ trade, then $m(T) \neq 2k - 1$.*

PROOF: Assume T is a Steiner $(k, 2)$ trade and $m(T) = 2k - 1$. Then $r(T) = 2$ by Lemma 4.15. By Lemma 4.16 there exists two elements a and b each occurring in at least $(k-1)$ blocks of T_1 in which the other element does not occur. We show that a and b are contained in precisely $k-1$ blocks each. Suppose there is a block in T_1 that contains both a and b. Any other element of this block must occur in at least one other block by Lemma 4.12 but all the remaining blocks contain either a or b which contradicts the fact that T is Steiner. Also, one of a or b is not contained in k blocks by Lemma 4.17. Thus there is a block $B \in T_1$ containing neither a nor b, and a and b are each contained in precisely $k-1$ blocks.

Consider any element α other than a or b; we show that for $k > 4$, if α occurs in a block with a, then it must also occur in a block with b. For if not, then $r_\alpha = 2$ and $\alpha \in B$. By Lemma 4.16 there exists element c contained in B such that c is contained in at least $(k-1)$ blocks. We have shown that c cannot equal a or b and so there are three distinct elements a, b, c each occurring in at least $k-1$ blocks. Further, there is no block containing both a and b and so the number of blocks in T_1 is at least $3(k-1)-2 > 2k-1$ for $k > 4$. This is a contradiction and so we conclude that any element occurring in a block with a occurs in a block with b and conversely.

Now, consider the k elements of the block B. Each of these elements occurs in at least one more block of T_1 and so each of these elements occurs in a block with a. But a is contained in exactly $k-1$ blocks and so one of the pairs of elements in B is repeated in T_1 contradicting the fact that T_1 is a Steiner $(k, 2)$ trade.

That the result holds when $k = 3$ or 4 is known, see [7], and this completes the proof. □

LEMMA 4.20: *Suppose* $k = 3, 4$ *or* 5 *and let* $T = T_1 - T_2$ *be a Steiner* $(k, 2)$ *trade with* $m(T) = 2k$.
(1) *If* $k = 3$, *then either* $T_1 = xA^1 + yA^2$ *and* $T_2 = xA^2 + yA^1$, *or* $T_1 = xA^1 + yA^2 + zA^3$ *and* $T_2 = xA^2 + yA^3 + zA^1$, *where* A^1 *and* A^2 (*respectively* A^1, A^2 *and* A^3) *are solely* 1-*balanced and* $x, y, z \notin F(A^1)$;
(2) *If* $k = 4$, *then there are at least two structures for* T. *One of these has* $T_1 = xA^1 + yA^2$ *and* $T_2 = xA^2 + yA^1$, *where* A^1 *and* A^2 *are solely* 1-*balanced and* $x, y \notin F(A^1)$;
(3) *If* $k = 5$, *then* $T_1 = xA^1 + yA^2$ *and* $T_2 = xA^2 + yA^1$, *where* A^1 *and* A^2 *are solely* 1-*balanced and* $x, y \notin F(A^1)$.

PROOF: If $k = 3$ and $m(T) = 6$, then all the elements of $F(T)$ have multiplicity two or three, by Lemma 4.14, and it is straightforward to check that T must be as claimed. The trades T^a and T^b in Example 4.4 illustrate these two structures.

For $k = 4$, it is easy to construct the trade based on solely 1-balanced families. For another structure, consider the fact that no element in

$$\begin{aligned} T &= +045a + 069b + 167a + 158b + 289a + 247b + 3468 + 3579 \\ &\quad -046a - 059b - 158a - 167b - 279a - 248b - 3457 - 3689 \end{aligned}$$

has multiplicity four.

See Appendix C for a proof of the case where $k = 5$. □

THEOREM 4.21: *Suppose* $k > 5$, $T = T_1 - T_2$ *is a Steiner* $(k, 2)$ *trade, and* $m(T) = 2k$. *Then* $T_1 = xA^1 + yA^2$ *and* $T_2 = xA^2 + yA^1$, *where* A^1 *and* A^2 *are solely* 1-*balanced and* $x, y \notin F(A^1)$.

Proof: If $m(T) = 2k$ then, by Lemma 4.15, $r(T) = 2$. For $k > 5$, $m(T) = 2k < 4k - 10$ and by Lemma 4.16 there exist at least $k - 1$ blocks in T_1 which contain x but not y and at least $k - 1$ blocks which contain y but not x. Suppose there is a block B in T_1 which contains both x and y. The remaining $k - 2$ elements of B must occur at least twice in T_1. All, except possibly one, blocks of T_1 contain either x or y and so the remaining $k - 2$ elements cannot occur again in T_1 without repeating a pair. Thus x and y do not occur in a block together.

Suppose that B is a block in T_1 which contains neither x nor y. We show that the multiplicity of any element c which is contained in B is at least three. Suppose $r_c = 2$. Then by Lemma 4.16, there exists a distinct element, say z, such that z is in B and z is contained in at least $k - 1$ blocks of T_1. This would imply that $m(T) \geq 3(k-1) - 2 = 3k - 5$. Thus $m(T) > 2k$ for $k > 5$, which is a contradiction, so $r_c \geq 3$.

The only possible values for (r_x, r_y) are $(k-1, k-1)$, $(k, k-1)$, $(k-1, k)$ or (k, k), and we consider each of these cases separately.

Case $(k-1, k-1)$: Consider the two blocks C and D which contain neither x nor y. One of the k elements of C, say a, is not contained in any of the $k - 1$ blocks in which y occurs (or else a pair is repeated in the blocks of T_1). But we have already shown that $r_a \geq 3$. This implies that $a \in D$ and in a block containing x. Similarly, one of the elements of C, say b, is not contained in any of the $k - 1$ blocks in which x occurs. But $r_b \geq 3$, which implies $b \in D$. The case $a = b$ is disallowed by our construction and the pair $\{a, b\}$ is repeated in the blocks C and D which is a contradiction.

Case $(k, k-1)$ or $(k-1, k)$: Without loss of generality, we show only that the case $(k, k-1)$ is impossible. Consider the block C which contains neither x nor y. One of the k elements of C, say a, is not contained in any of the $k - 1$ blocks in which y occurs (or else a pair is repeated in the blocks of T_1). As $r_a \geq 3$, this would imply that a is contained in two blocks which contain x and the pair $\{a, x\}$ is repeated, a contradiction.

Case (k, k): This case must hold. Now $r_x + r_y = m(T)$ and the structure of T follows from Lemma 4.17. □

We now use the solely 1-balanced families $\{R, C, F\}$ and $\{\underline{R}_r, \underline{C}_r\}$ of Section 4.2 to construct Steiner $(k, 2)$ trades for all the required volumes. We start with an example, to illustrate the technique.

Example 4.22: Recall Example 4.10. Now let

$$\begin{aligned} T^a &= +xR + yF - xF - yR, \\ T^b &= -x\underline{R}_1 - z\underline{C}_1 + x\underline{C}_1 + z\underline{R}_1, \end{aligned}$$

where $x, y, z \notin F(R \cup \underline{R}_1)$. Then

$$\begin{aligned} T^a + T^b &= +x123 + y159 + y267 + y348 \\ &\quad +x\underline{1}47 + x\underline{2}58 + x\underline{3}69 + z\underline{123} + z456 + z789 \\ &\quad -x159 - x267 - x348 - y123 - y456 - y789 \\ &\quad -x\underline{123} - z\underline{1}47 - z\underline{2}58 - z\underline{3}69, \end{aligned}$$

is a Steiner $(4, 2)$ trade of volume ten. Note how $+x456$ and $+x789$ are in T^a and $-x456$ and $-x789$ are in T^b and that these blocks cancel in $T^a + T^b$. □

The straightforward construction of Example 4.22 will be modified to prove our existence results. In Lemmas 4.23 and 4.24 we choose the foundations of T^a and T^b so that $T^a + T^b$ is a Steiner trade of the required volume.

Lemma 4.23: *There exists a Steiner* $(k, 2)$ *trade of volume* $2(k-1) + 2r$ *for each* $r = 0, 1, \ldots k-1$.

Proof: Let $T^a = T^a_1 - T^a_2 = +xR + yF - xF - yR$ and $T^b = T^b_1 - T^b_2 = +x\underline{C}_r + y\underline{R}_r - x\underline{R}_r - y\underline{C}_r$ with distinct $x, y \notin F(R \cup \underline{R}_r)$. That T^a, T^b and $T^a + T^b$ are trades follows from Lemmas 3.3 and 4.8. It remains to show that $T^a + T^b$ is a Steiner trade of the required volume.

Any pair of elements in xR that occurs more than once in $T^a_1 + T^b_1$ must occur in $y\underline{R}_r$ or $x\underline{C}_r$. Any pair of elements in yF that occurs more than once in $T^a_1 + T^b_1$ must occur in $y\underline{R}_r$. However, the blocks containing such pairs cancel in the addition of T^a and T^b; either in the $k - 1 - r$ blocks in xR which cancel with $x\underline{R}_r$ or in the $k - 1 - r$ blocks in $y\underline{R}_r$ which cancel with yR. Thus $T^a + T^b$ is a Steiner $(k, 2)$ trade of volume

$$\begin{aligned} m(T^a + T^b) &= 4(k-1) - |xR \cap x\underline{R}_r| - |yR \cap y\underline{R}_r| \\ &= 4(k-1) - 2(k-1-r) \\ &= 2(k-1) + 2r, \end{aligned}$$

as required. □

LEMMA 4.24: *There exists a Steiner* $(k,2)$ *trade of volume* $3(k-1)+r$ *for each* $r = 0, 1, \dots k-1$.

PROOF: Let $T^a = +xR + yF - xF - yR$ and $T_b = +x\underline{C}_r + z\underline{R}_r - x\underline{R}_r - z\underline{C}_r$, with distinct $x, y, z \notin F(R \cup \underline{R}_r)$. Then, as in the proof of Lemma 4.23, $T^a + T^b$ is a Steiner $(k,2)$ trade of volume

$$4(k-1) - |xR \cap x\underline{R}_r| = 4(k-1) - (k-1-r) = 3(k-1) + r. \qquad \Box$$

LEMMA 4.25: *There exists a Steiner* $(k,2)$ *trade of volume* $4(k-1)+r$ *for each* $r = 0, 1, \dots k-1$.

PROOF: First note that by Lemma 4.23, $2(k-1) \in \mathcal{S}_1(k,2)$, as are all even values between $2(k-1)$ and $3(k-1)$ inclusive. Thus all values between $4(k-1)$ and $5(k-1)$ inclusive are in $\mathcal{S}_1(k,2)$ by Lemma 4.3. This completes the proof. $\Box$

THEOREM 4.26: *If* $m \geq 3(k-1)$, *then there exists a Steiner* $(k,2)$ *trade of volume* m.

PROOF: We have shown that $2(k-1) \in \mathcal{S}_1(k,2)$ and that $\{3k-3, 3k-2, \dots, 5k-5\} \subseteq \mathcal{S}_1(k,2)$. As $\mathcal{S}_1(k,2)$ is closed under addition, the result follows. $\Box$

It remains to determine whether the odd integers between $2k+1$ and $3k-4$ inclusive are in $\mathcal{S}_1(k,2)$, for $k \geq 5$. We complete the cases $k = 5$ and 6, and leave one volume unresolved for the case $k = 7$.

THEOREM 4.27: $\mathcal{S}_1(5,2) = \{0, 8, 10, 12, 13, 14, \dots\}$.

PROOF: By the results of this section, the only unresolved volume for $\mathcal{S}_1(5,2)$ is eleven. A proof that $11 \notin \mathcal{S}_1(5,2)$ is given in Appendix C. $\Box$

THEOREM 4.28: $\mathcal{S}_1(6,2) = \{0, 10, 12, 14, 15, 16, \dots\}$.

PROOF: By the results of this section, the only unresolved volume for $\mathcal{S}_1(6,2)$ is thirteen. A computer-assisted proof that $13 \notin \mathcal{S}_1(6,2)$ is given in Appendix C. (There is also a theoretical proof in Section 4.6.) $\Box$

THEOREM 4.29: $\mathcal{S}_1(7,2) \supseteq \{0, 12, 14, 15, 16, 18, 19, 20, \dots\}$ *and* $\overline{\mathcal{S}}_1(7,2) \supseteq \{1, 2, \dots, 11, 13\}$. *The existence of a Steiner* $(7,2)$ *trade of volume seventeen is unresolved.*

PROOF: By the results of this section, the only unresolved volumes for $\mathcal{S}_1(7,2)$ are fifteen and seventeen. A Steiner $(7,2)$ trade $T = T_1 - T_2$ of volume fifteen is given in Figure 4.1. (This trade is discussed in Section 4.6.) $\Box$

FIGURE 4.1: A Steiner $(7,2)$ trade of volume 15

$$T_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 13 & 14 & 15 & 16 & 17 & 18 \\ 1 & 7 & 13 & 19 & 20 & 21 & 22 \\ 1 & 8 & 14 & 23 & 24 & 25 & 26 \\ 2 & 7 & 14 & 27 & 28 & 29 & 30 \\ 2 & 8 & 13 & 31 & 32 & 33 & 34 \\ 3 & 9 & 15 & 19 & 23 & 27 & 31 \\ 3 & 10 & 16 & 20 & 24 & 28 & 32 \\ 4 & 9 & 16 & 21 & 25 & 29 & 33 \\ 4 & 10 & 15 & 22 & 26 & 30 & 34 \\ 5 & 11 & 17 & 19 & 24 & 29 & 34 \\ 5 & 12 & 18 & 20 & 23 & 30 & 33 \\ 6 & 11 & 18 & 21 & 26 & 27 & 32 \\ 6 & 12 & 17 & 22 & 25 & 28 & 31 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 1 & 2 & 7 & 8 & 13 & 14 \\ 0 & 3 & 4 & 9 & 10 & 15 & 16 \\ 0 & 5 & 6 & 11 & 12 & 17 & 18 \\ 1 & 3 & 5 & 19 & 20 & 23 & 24 \\ 1 & 4 & 6 & 21 & 22 & 25 & 26 \\ 2 & 3 & 6 & 27 & 28 & 31 & 32 \\ 2 & 4 & 5 & 29 & 30 & 33 & 34 \\ 7 & 9 & 11 & 19 & 21 & 27 & 29 \\ 7 & 10 & 12 & 20 & 22 & 28 & 30 \\ 8 & 9 & 12 & 23 & 25 & 31 & 33 \\ 8 & 10 & 11 & 24 & 26 & 32 & 34 \\ 13 & 15 & 17 & 19 & 22 & 31 & 34 \\ 13 & 16 & 18 & 20 & 21 & 32 & 33 \\ 14 & 15 & 18 & 23 & 26 & 27 & 30 \\ 14 & 16 & 17 & 24 & 25 & 28 & 29 \end{bmatrix}$$

4.4 Steiner $(k, 2)$ trades, volume $2k + 1$: preliminaries

We have proved that Steiner (k, t) trades of volume $2k - 1$ do not exist. The odd volume $2k + 1$, when this is less than $3(k - 1)$ (that is, when $k > 4$), has only been solved for $k = 5, 6, 7$. We complete the solution of this case, and show how the volume fifteen Steiner $(7, 2)$ trade was constructed. Our solution depends essentially on the following lemma, which, in turn, depends on Lemmas 4.16 and 4.17.

LEMMA 4.30: *Suppose $k > 5$ and $T = T_1 - T_2$ is a Steiner $(k, 2)$ trade. If $m(T) = 2k + 1$ and $\alpha \in F(T)$, then $r_\alpha \neq 2$.*

PROOF: Suppose there exists $\alpha \in F(T)$ with $r_\alpha = 2$. For $k > 5$, $m(T) = 2k + 1 < 4k - 10$. By Lemma 4.16 there exist elements x, y such that at least $k - 1$ blocks of T_1 contain x but not y and at least $k - 1$ blocks of T_1 contain y but not x.

(i) Suppose there is a block B in T_1 which contains both x and y. The remaining $k - 2$ elements of B must occur at least twice in T_1. All, except possibly two, blocks of T_1 contain either x or y and so the remaining $k - 2$ elements of B cannot occur again in T_1 without repeating a pair for $k > 5$. Thus x and y do not occur in a block together.

(ii) Suppose that B is a block in T_1 which contains neither x nor y. We show that the multiplicity of any element e which is contained in B is at least three. Suppose that $r_e = 2$. Then by Lemma 4.16, there exists a new element z such that z is in B and z is contained in at least $k - 1$ blocks of T_1. Either there exists a fourth element contained in at least $k-1$ blocks of T_1, which is clearly impossible, or at least one of the pairs $\{x,z\}$ or $\{y,z\}$ does not occur in the blocks of T_1 by (i). Thus $m(T) \geq 3(k-1)-1 = 3k-4 > 2k+1$ for $k > 5$, which is a contradiction. Hence $r_e \geq 3$.

Without loss of generality suppose $r_x \geq r_y$. By (i), the only possible values for (r_x, r_y) are $(k-1,k-1)$, $(k,k-1)$, $(k+1,k-1)$, $(k+2,k-1)$, (k,k) and $(k+1,k)$. We shall now use (ii) and other techniques to show that all these cases lead to a contradiction. The most troublesome case is dealt with first.

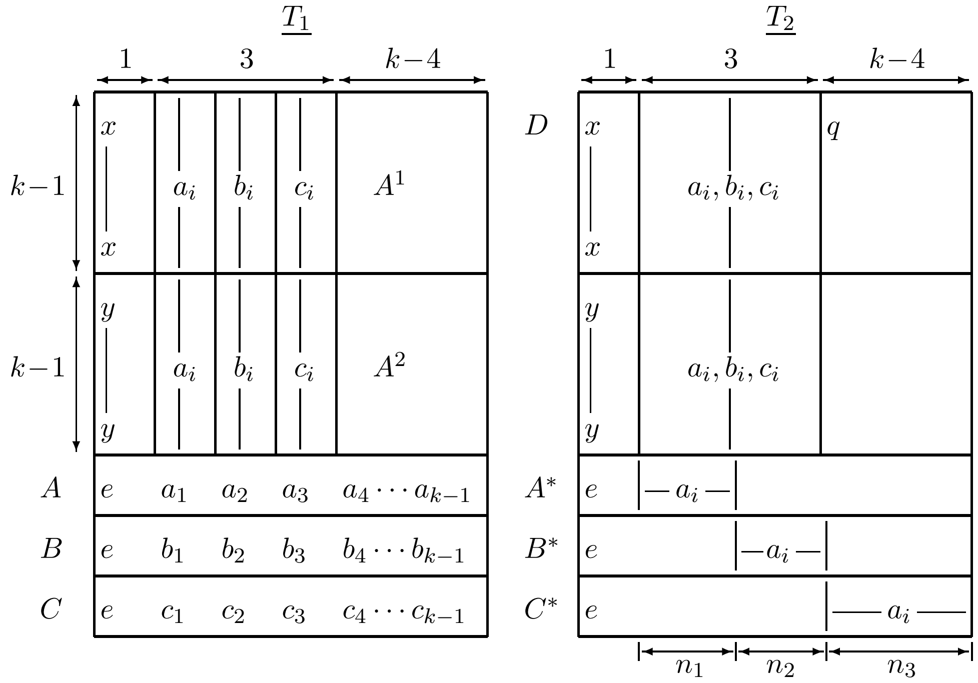
Case $(k-1,k-1)$: Consider the k elements of any of the three blocks which contain neither x nor y. At least one of these elements, e say, is not contained in a block with y since T is Steiner and only $k-1$ blocks contain y; that is, $r_{\{e,y\}} = 0$.

Subcase (a): Suppose that e is not contained in a block with x, that is $r_{\{e,x\}} = r_{\{e,y\}} = 0$. It is immediate from (ii) that $r_e = 3$. Let $A = \{e, a_1, a_2, \ldots, a_{k-1}\}$, $B = \{e, b_1, b_2, \ldots, b_{k-1}\}$ and $C = \{e, c_1, c_2, \ldots, c_{k-1}\}$ represent the three blocks which contain e. It follows that each of x, y, e, a_i, b_j, c_l is distinct. Each of the elements a_i, b_i, c_i must occur in at least three blocks of T_1. Thus T_1 can be represented as in Figure 4.2 where A^1 and A^2 are solely 1-balanced. Note that each of the blocks in which x or y occurs contains precisely three elements from $A \cup B \cup C \setminus \{e\}$. Now consider T_2. Let A^*, B^* and C^* represent the three blocks of T_2 which contain e. To balance pairs containing e in T_1 and T_2, it is immediate that $A \cup B \cup C \setminus \{e\} = A^* \cup B^* \cup C^* \setminus \{e\}$. By symmetry, each of the blocks in T_2 in which x or y occurs contains precisely three elements from $A^* \cup B^* \cup C^* \setminus \{e\}$. Let $n_1 = |A^* \cap A \setminus \{e\}|, n_2 = |B^* \cap A \setminus \{e\}|$ and $n_3 = |C^* \cap A \setminus \{e\}|$. Note that $n_1 + n_2 + n_3 = k-1$.

We show that for $k > 5$, one of n_1, n_2 or $n_3 = k-1$ and thus $T_1 \cap T_2 \neq \emptyset$, which is a contradiction. We count pairs of the form $\{a_i, a_j\}$. Consider a block D in T_2 which contains x. It has been shown that precisely three elements from $A \cup B \cup C \setminus \{e\}$ are contained in D.

We first show that these three elements cannot all be from the same block, A say. As $k > 5$, there exists a point q in D which is in a $(k-4)$-subset of A^1. As $q \in F(A^1)$, $r_q = 2$. Thus the three elements of A occur in a pair with q in T_2. However, any three elements from $A \setminus \{e\}$ occur in distinct blocks of $T_1 \setminus \{A\}$. But $r_q = 2$ and hence three elements from A cannot all be contained in block D.

FIGURE 4.2: The trade $T_1 - T_2$ of subcase (a)



So at most two of the a_i occur in a block with x in T_2. An a_i can occur in at most one block with x in T_2 and thus there are at most $(k-1)/2$ pairs from A in the blocks containing x in T_2. Similarly, there are at most $(k-1)/2$ pairs from A in the blocks containing y in T_2.

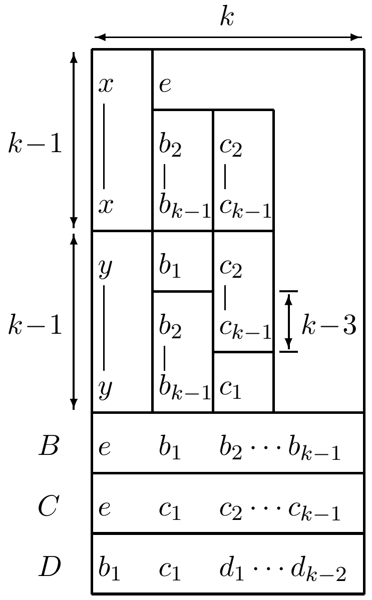
Suppose that the exact number of pairs from A in the blocks of T_2 containing x or y equals s. The total number of pairs from A in T_1 equals $\binom{k-1}{2}$. However the number of pairs from A in T_1 and T_2 is equal as T is a trade. This, together with the constraints $s \leq k-1$ and $n_1 + n_2 + n_3 = k-1$, yields

$$\begin{array}{rrcl} & \binom{k-1}{2} & = & s + \binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} \\ \Rightarrow & (k-1)^2 - 2s & = & n_1^2 + n_2^2 + n_3^2 \\ \Rightarrow & (k-1)^2 - 2(k-1) & \leq & n_1^2 + n_2^2 + n_3^2 \\ \Rightarrow & (k-2)^2 - 1 & \leq & n_1^2 + n_2^2 + n_3^2. \end{array} \qquad (*)$$

Recall that $n_i \neq k-1$ for $i = 1, 2, 3$. If $\{n_1, n_2, n_3\} \neq \{k-2, 1, 0\}$, then $n_1^2 + n_2^2 + n_3^2 \leq (k-3)^2 + 2^2$ which for $k > 5$ clearly contradicts $(*)$. So $\{n_1, n_2, n_3\} = \{k-2, 1, 0\}$, which forces $s = k-2$. But for these values of the n_i, the maximum value of s is seen to be two. This implies that $k \leq 4$, which contradicts $k > 5$. Hence $n_i = k-1$ for some i, which completes the proof of Subcase (a).

Subcase (b): For this case we suppose that the element e is contained in a block with x, that is $r_{\{e,x\}} = 1$ and $r_{\{e,y\}} = 0$. If $r_e = 4$, consider any block B which

FIGURE 4.3: The trade T_1 of subcase (b)



contains e but neither x nor y. By (ii), the $k-1$ elements of B other than e must each have multiplicity at least three. However, this is impossible as there are only $k-2$ blocks which contain x and not e.

Thus $r_e = 3$ and the two blocks of T_1 that contain neither x nor y can be written, without loss of generality, as $B = \{e, b_1, b_2, \ldots, b_{k-1}\}$ and $C = \{e, c_1, c_2, \ldots, c_{k-1}\}$. Each of the elements b_i, c_i occur in at least three blocks of T_1. By considering the fact that there are only $k-2$ blocks occurring with x and not with e, it is easily seen that the block D in T_1 which does not contain x, y or e must contain exactly one b_i and one c_i (say b_1 and c_1). Moreover, as each of the b_i, c_i has multiplicity at least three, b_1 and c_1 must both occur in (different) blocks with y. Without loss of generality, let $D = \{b_1, c_1, d_1, d_2, \ldots, d_{k-2}\}$ and note that $d_i \neq x, y, e, b_j, c_j$ for $i = 1, \ldots, k-2$, $j = 1, \ldots, k-1$. We thus have the representation of T_1 in Figure 4.3.

Again, each of the d_i must have multiplicity three in T_1. There are only $k-3$ blocks containing y in which the d_i can be placed as two of the blocks containing y contain b_1 or c_1. Thus, at least one d_i has multiplicity less than three, which is a contradiction.

Case $(k, k-1)$: There must be at least $k-1$ elements which occur in a block with x but do not occur in a block with y. Each of these $k-1$ elements must occur in a block that contains neither x nor y. As the multiplicity of such elements must be at least three, this implies that each of these elements is contained in the two blocks which contain neither x nor y. A pair of elements is thus repeated contradicting the fact that T is Steiner.

Case $(k+1, k-1)$: There are at least $2(k-1)$ elements which occur in a block with x but do not occur in a block with y. Each of these elements must occur twice, which is impossible as there is only one remaining block in which they can be placed.

Case $(k+2, k-1)$: This is not possible by Lemma 4.17.

Case (k, k): Consider the block B in T_1 which contains neither x nor y. Each of the elements in B must occur in exactly three blocks of T_1. Hence, each of these elements must occur in exactly three blocks of T_2. In T_2, they can occur in precisely one block with x, one block with y and in the block B^* which contains neither x nor y. This implies that $B^* = B$, contradicting the fact that T is a trade.

Case $(k+1, k)$: This is not possible by Lemma 4.17. □

4.5 Regular and linked trades

We now introduce the concepts of *regular* and *linked* trades. It may well be interesting to study regular and linked trades in their own right, but here we do only what is necessary for the problem at hand. To this end, we have restricted the following definition to Steiner $(k, 2)$ trades.

DEFINITION 4.31: *Let $T = T_1 - T_2$ be a Steiner $(k, 2)$ trade. We say that:*
(1) *T is* ***regular*** *of* ***degree*** *d if $r_x = d$ for all $x \in F(T)$;*
(2) *T is* ***linked*** *if each pair of blocks from T_1, and each pair of blocks from T_2, intersect in precisely one element.*

An example of a linked and regular Steiner $(7, 2)$ trade of volume 15 was exhibited in Figure 4.1, while a linked but non-regular Steiner $(4, 2)$ trade of volume 9 is given in Example 4.36. A regular, but non-linked, Steiner $(3, 2)$ trade can be obtained by summing Pasch trades on disjoint foundations.

In this and the next two sections we show, amongst other things, how these trades were constructed. We now present some elementary results regarding the volumes of regular Steiner trades.

LEMMA 4.32: *Suppose that $T = T_1 - T_2$ is a Steiner $(k, 2)$ regular trade of degree r. Then $m(T) \geq k(r-1) + 1$.*

PROOF: Consider any block $\{a_1, a_2, \ldots, a_k\}$ in T_1. Each of the elements $a_1, a_2, \ldots, a_k$ occurs $r-1$ more times in the blocks of T_1. As no pair of these elements can occur in more than one block of T_1, there are at least $k(r-1)+1$ blocks in T_1. □

LEMMA 4.33: *Suppose that $T = T_1 - T_2$ is a Steiner $(k, 2)$ regular trade of degree r and minimum volume $m(T) = k(r-1)+1$. Then T is linked.*

PROOF: Let $f = f(T)$ and $m = m(T)$. Let I be the total number of pairs of intersecting blocks in, say, T_1. Then $I = \binom{r}{2} f$. However, $f = mk/r$, and some algebra yields $I = \binom{m}{2}$. Thus every pair of blocks of T_1, and of T_2, intersects in one element and T is linked. □

In Section 4.3, we showed that $2k+1 \in \mathcal{S}_1(k, 2)$ if $k = 3, 4$ or 7 and $2k+1 \notin \mathcal{S}_1(k, 2)$ if $k = 5$ or 6. For the latter two cases, we introduced the concept of a swap matrix. We now present an alternative method, which allows us to extend these results, by showing that Steiner $(k, 2)$ trades of volume $2k+1$, $k > 5$, are necessarily regular of degree three and hence linked.

LEMMA 4.34: *Suppose that $k > 5$ and T is a $(k, 2)$ Steiner trade of volume $2k+1$. Then T is a regular trade of degree three.*

PROOF: By Lemma 4.30, no element in T has multiplicity two. Assume there exists an element x of multiplicity r_x strictly greater than three. Let B be any block in T_1 containing x. The $k-1$ elements of B other than x each have multiplicity at least three. However, other than in B, no pair of these elements can occur in a block of T_1. Also, none of these elements can appear in any other block of T_1 that contains x. Thus $m(T) \geq r_x + 2(k-1) > 2k+1$, a contradiction. Hence, every element in T has multiplicity three. □

Note that the previous lemma immediately implies that if $k > 5$ and $2k+1 \in \mathcal{S}_1(k, 2)$, then $k \equiv 0, 1 \pmod 3$.

LEMMA 4.35: *Suppose that $T = T_1 - T_2$ is a Steiner $(k, 2)$ regular trade of degree r and $m(T) = k(r-1)+1$. Then T_1 is the dual of a 2-$(v, r, 1)$ design with $v = k(r-1)+1$.*

PROOF: By Lemma 4.33, T_1 is necessarily linked. Let T_1^* be the dual of T_1. Then each element of T_1^* is contained in k blocks. Each block of T_1^* contains r elements. Finally, as T_1 is linked, each pair of elements of T_1^* is contained in precisely one block of T_1^*. Thus T_1^* is a 2-$(v, r, 1)$ design, where $v = m(T_1) = m(T)$. □

Remark: In the general case, the dual of a linked Steiner $(k, 2)$ trade is an (r, λ)-design with $r = k$ and $\lambda = 1$; see [114] for a discussion of (r, λ)-designs.

Example 4.36: A non-regular linked Steiner $(4, 2)$ trade of volume nine is

$$\begin{aligned} T = T_1 - T_2 \;=\; & +0134 + 0256 + 1278 + 239a + 24bc + 159b + 16ac + 079c + 08ab \\ & -0234 - 1256 - 0178 - 139a - 14bc - 059b - 06ac - 279c - 28ab. \end{aligned}$$

Note that the dual of each of T_1 and T_2 is a $(4, 1)$-design on nine elements. For example, the collection of blocks of the dual of T_1 is

$$\{0178, 0256, 1234, 03, 04, 15, 16, 27, 28, 357, 368, 458, 467\}. \qquad \square$$

4.6 Steiner $(k, 2)$ trades, volume $2k + 1$: solution

We now restrict our attention to finding Steiner $(k, 2)$ trades of volume $2k + 1$. To do this we will use some elementary graph theory.

Definition 4.37: *Let D be a design. Then the* ***block intersection graph*** *of D ($BIG(D)$) is the graph whose vertices are the blocks of D, with two vertices being adjacent if and only if they have a point in common.*

Definition 4.38: *K_k is the complete graph on k points. A subgraph isomorphic to K_k is known as a* ***k-clique****.*

Supppose that $T = T_1 - T_2$ is a Steiner $(k, 2)$ regular trade of degree three and volume $v = 2k + 1$. We know from Theorem 4.35 that T_1 and T_2 are the duals of 2-$(v, 3, 1)$ designs, D_1 and D_2 say. Label the blocks of D_1 and D_2 with the appropriate elements of $F(T)$. The pair of elements $\{a, b\}$ occurring in a block of T_1 is equivalent to an edge joining vertices a and b in $BIG(D_1)$. However, as T is a trade, any pair of elements occurring in the blocks of T_1 occurs in the blocks of T_2. Hence we deduce that $BIG(D_1) = BIG(D_2)$ and T_1 and T_2 are two disjoint k-clique decompositions of $BIG(D_1)$.

Now consider k-clique decompositions of $BIG(D_1)$. A k-clique of $BIG(D_1)$ corresponds to a collection S of k blocks of D_1 that have non-empty pairwise intersection. Suppose that there is not an element common to all k blocks of S. It is simple to see that the cardinality of S is at most 7 (in which case S is a Fano plane). Thus, for $k > 7$, a k-clique of $BIG(D_1)$ corresponds to a collection of k blocks containing a common point and the k-clique decomposition of $BIG(D_1)$ is unique. We thus have the following result.

Figure 4.4: The unique 2-$(9,3,1)$ design

a :	123	d :	147	g :	159	j :	168
b :	456	e :	258	h :	483	k :	249
c :	789	f :	369	i :	267	l :	357

Lemma 4.39: *For $k > 7$, $2k+1 \notin \mathcal{S}_1(k,2)$.* □

It is known that for Steiner 2-$(v,k,1)$ designs D_1 and D_2, with $v > k(k^2-2k+2)$, if $BIG(D_1)$ is isomorphic to $BIG(D_2)$, then D_1 is isomorphic to D_2 [30]. For $STS(v)$, this result also holds when $v \leq 15$. The only difficult case is when $v = 15$, and this case was done in, for example, [12].

Lemma 4.40: *For $5 < k \leq 7$, the following are equivalent:*
(1) *The existence of a Steiner $(k,2)$ trade T of volume $2k+1$;*
(2) *The existence of a* 2-$(2k+1,3,1)$ *design D with $BIG(D)$ possessing two distinct k-clique decompositions.*

Proof: From the preceeding discussion, it is immediate that (1) implies (2). Certainly (2) implies the existence of a Steiner $(k,2)$ trade T of volume no more than $2k+1$ and with element multiplicities no greater than three. The structures of Steiner trades of volumes $2k-2$ and $2k$ have been determined in Theorem 4.18, Lemma 4.20 and Theorem 4.21. Trades of such volumes contain two distinct elements of multiplicity at least $k-1 > 4$, so T must be as claimed. □

Although the previous lemma will be used to discuss the case $5 < k \leq 7$, we can illustrate the general method by constructing a Steiner $(4,2)$ trade of volume nine which is regular of degree three and linked.

Example 4.41: Let D be the unique 2-$(9,3,1)$ shown in Figure 4.4, where the letters represent block labels. $BIG(D)$ is the complete multipartite graph with parts $\{a,b,c\}$, $\{d,e,f\}$, $\{g,h,i\}$ and $\{j,k,l\}$. One 4-clique decomposition of $BIG(D)$ is

$$T_1 = adgj + aeik + afhl + bdhk + begl + bfij + cdil + cehj + cfgk.$$

It is easy to see that the permutation (abc) applied to the block labelling of D induces the 4-clique decomposition

$$T_2 = bdgj + beik + bfhl + cdhk + cegl + cfij + adil + aehj + afgk.$$

$T = T_1 - T_2$ is then a Steiner $(4,2)$ trade of volume nine which is regular of degree three and linked. □

Figure 4.5: The blocks of P, in rows of Fano planes

F_1:	**1** 1 2 5	**4** 4 5 8	**15** 15 1 4	**16** 1 8 10	**23** 15 2 8	**25** 2 4 10	**31** 15 5 10
F_2:	**2** 2 3 6	**5** 5 6 9	**1** 1 2 5	**17** 2 9 11	**24** 1 3 9	**26** 3 5 11	**32** 1 6 11
F_3:	**3** 3 4 7	**6** 6 7 10	**2** 2 3 6	**18** 3 10 12	**25** 2 4 10	**27** 4 6 12	**33** 2 7 12
F_4:	**4** 4 5 8	**7** 7 8 11	**3** 3 4 7	**19** 4 11 13	**26** 3 5 11	**28** 5 7 13	**34** 3 8 13
F_5:	**5** 5 6 9	**8** 8 9 12	**4** 4 5 8	**20** 5 12 14	**27** 4 6 12	**29** 6 8 14	**35** 4 9 14
F_6:	**6** 6 7 10	**9** 9 10 13	**5** 5 6 9	**21** 6 13 15	**28** 5 7 13	**30** 7 9 15	**31** 5 10 15
F_7:	**7** 7 8 11	**10** 10 11 14	**6** 6 7 10	**22** 7 14 1	**29** 6 8 14	**16** 8 10 1	**32** 6 11 1
F_8:	**8** 8 9 12	**11** 11 12 15	**7** 7 8 11	**23** 8 15 2	**30** 7 9 15	**17** 9 11 2	**33** 7 12 2
F_9:	**9** 9 10 13	**12** 12 13 1	**8** 8 9 12	**24** 9 1 3	**16** 8 10 1	**18** 10 12 3	**34** 8 13 3
F_{10}:	**10** 10 11 14	**13** 13 14 2	**9** 9 10 13	**25** 10 2 4	**17** 9 11 2	**19** 11 13 4	**35** 9 14 4
F_{11}:	**11** 11 12 15	**14** 14 15 3	**10** 10 11 14	**26** 11 3 5	**18** 10 12 3	**20** 12 14 5	**31** 10 15 5
F_{12}:	**12** 12 13 1	**15** 15 1 4	**11** 11 12 15	**27** 12 4 6	**19** 11 13 4	**21** 13 15 6	**32** 11 1 6
F_{13}:	**13** 13 14 2	**1** 1 2 5	**12** 12 13 1	**28** 13 5 7	**20** 12 14 5	**22** 14 1 7	**33** 12 2 7
F_{14}:	**14** 14 15 3	**2** 2 3 6	**13** 13 14 2	**29** 14 6 8	**21** 13 15 6	**23** 15 2 8	**34** 13 3 8
F_{15}:	**15** 15 1 4	**3** 3 4 7	**14** 14 15 3	**30** 15 7 9	**22** 14 1 7	**24** 1 3 9	**35** 14 4 9

We now complete our search for k-clique decompositions for the case $5 < k \leq 7$. For $k = 6$, it is simple to verify that the 6-clique decompositions of the block intersection graphs of the two non-isomorphic 2-$(13, 3, 1)$ designs are unique; in fact, the only 6-cliques in these graphs correspond to an element of the design.

The 7-cliques in the block intersection graph of a 2-$(15, 3, 1)$ design are either a v-**type**, corresponding to an element of the design, or an f-**type**, corresponding to a Fano plane subdesign. A v-type and an f-type clique intersect each other in either 0 or 3 vertices. However, a counting argument, analagous to that used in Lemma 4.33, shows that the k-cliques of a decomposition of $BIG(D)$ for a 2-$(2k + 1, 3, 1)$ design D must intersect each other in precisely one vertex. Thus, if there are two distinct 7-clique decompositions of the block intersection graph of a 2-$(15, 3, 1)$ design D, then D necessarily contains at least 15 Fano planes. There is only one such 2-$(15, 3, 1)$ design; see, for instance, [88]. This is the design associated with $PG(3, 2)$. The blocks of this design, P say, are listed in Figure 4.5 (block labels in bold). In $BIG(P)$, there are precisely 15 f-type cliques (intersecting each other in one vertex). Two distinct decompositions of $BIG(P)$ into v-type and f-type cliques are given in Figure 4.6; this trade is isomorphic to that given in the proof of Theorem 4.29.

Summarising the results of the previous two paragraphs in terms of volumes of Steiner $(k, 2)$ trades, we have that $13 \notin \mathcal{S}_1(6, 2)$ and that $15 \in \mathcal{S}_1(7, 2)$, with the latter trades obviously having a unique structure. Two structurally different Steiner $(4, 2)$ trades of volume nine were presented in Examples 4.36 and 4.41. Two struc-

FIGURE 4.6: The Steiner $(7, 2)$ trade of volume 15 from $BIG(P)$

V_1 :	1 12 15 16 22 24 32	F_1 :	1 4 15 16 23 25 31
V_2 :	2 13 1 17 23 25 33	F_2 :	2 5 1 17 24 26 32
V_3 :	3 14 2 18 24 26 34	F_3 :	3 6 2 18 25 27 33
V_4 :	4 15 3 19 25 27 35	F_4 :	4 7 3 19 26 28 34
V_5 :	5 1 4 20 26 28 31	F_5 :	5 8 4 20 27 29 35
V_6 :	6 2 5 21 27 29 32	F_6 :	6 9 5 21 28 30 31
V_7 :	7 3 6 22 28 30 33	F_7 :	7 10 6 22 29 16 32
V_8 :	8 4 7 23 29 16 34	F_8 :	8 11 7 23 30 17 33
V_9 :	9 5 8 24 30 17 35	F_9 :	9 12 8 24 16 18 34
V_{10} :	10 6 9 25 16 18 31	F_{10} :	10 13 9 25 17 19 35
V_{11} :	11 7 10 26 17 19 32	F_{11} :	11 14 10 26 18 20 31
V_{12} :	12 8 11 27 18 20 33	F_{12} :	12 15 11 27 19 21 32
V_{13} :	13 9 12 28 19 21 34	F_{13} :	13 1 12 28 20 22 33
V_{14} :	14 10 13 29 20 22 35	F_{14} :	14 2 13 29 21 23 34
V_{15} :	15 11 14 30 21 23 31	F_{15} :	15 3 14 30 22 24 35

turally different Steiner $(3, 2)$ trades of volume seven are

$$\begin{aligned} T^a &= +124 + 235 + 346 + 457 + 561 + 672 + 713 \\ &\quad -134 - 245 - 356 - 467 - 571 - 612 - 723, \\ T^b &= +123 + 145 + 624 + 635 + 178 + 928 + 947 \\ &\quad -135 - 623 - 645 - 128 - 147 - 924 - 978. \end{aligned}$$

Recall that $11 \notin \mathcal{S}_1(5, 2)$ (proved in Appendix C). Combining the results of this section with those of Section 4.3 establishes Theorem 4.1.

4.7 Results for $t > 2$

In this section we briefly consider the problem of determining $\mathcal{S}_1(k, t)$ when $t > 2$. We know that the volume of a (k, t) trade is at least 2^t. However, the volume of a smallest Steiner (k, t) trade grows rapidly with k, as the following result demonstrates.

THEOREM 4.42: *If $T = T_1 - T_2$ is a Steiner (k, t) trade, $t > 1$, then*

$$m(T) \geq 1 + \binom{k}{t-1}.$$

PROOF: Let B be a block in T_1. By Lemma 4.12, each $(t-1)$-subset of B must occur in at least one other block of T_1. If any two of these $(t-1)$-subsets occur together in a block other than B, then a t-subset is repeated in T_1, contradicting the fact that T is Steiner. The number of $(t-1)$-subsets in B is $\binom{k}{t-1}$ and the result follows. □

Our success in the theoretical investigation of Steiner $(k, 2)$ trades is due, in large part, to our ability to construct solely 1-balanced families. We do not have any general construction of solely t-balanced families for $t > 1$ and arbitrary block-size. However, two simple constructions are briefly illustrated in the following two subsections. We start with $\mathcal{S}_1(t+1, t)$. Since $\mathcal{S}_1(2, 1)$, $\mathcal{S}_1(3, 2)$ and $\mathcal{S}_1(4, 3)$ are known, we consider the case $t = 4$. We then investigate $\mathcal{S}_1(5, 3)$, the first unsolved case for $t = 3$.

4.7.1 Steiner $(t+1, t)$ trades, when $t = 4$

When $k = t+1$, Steiner trades themselves are a source of solely t-balanced families.

Lemma 4.43: *Let $T = T_1 - T_2$ be a Steiner $(t+1, t)$ trade. Then T_1 and T_2 are solely t-balanced.*

Proof: Any $(t+1)$-subset in T is a block, in T_1 say. Since T is Steiner and $T_1 \cap T_2 = \emptyset$, this block occurs only once in T_1 and cannot occur in T_2. $\square$

So Steiner $(4, 3)$ trades are solely 3-balanced families, and we can use Lemma 4.8 to construct Steiner $(5, 4)$ trades, giving $\mathcal{S}_1(5, 4) \supseteq \{16, 24, 28, 30, 32, \ldots\}$. Using Lemma 4.3 with these values gives $\mathcal{S}_1(5, 4) \supseteq \{31, 39, 43, 45, 47, \ldots\}$. It is straightforward, but tedious, to verify that the $(5, 4)$ trades of volumes 37 and 41 constructed in the proof of Lemma 3.36 are Steiner. We know that $\overline{\mathcal{S}}_1(5, 4) \supseteq \{1, \ldots, 15, 17, \ldots, 23, 25\}$ (Section 2.2 and Theorem 3.25), and we believe that $\overline{\mathcal{S}}_1(5, 4) \supseteq \{26, 27, 29, 33, 35\}$ (Conjectures 2.15 and 3.37).

Let $X = \{0, 16, 24, 28, 30, 31, 32, 34, 36, 37, 38, \ldots\}$. Then we have proved that $\mathcal{S}_1(5, 4) \supseteq X$, and we conjecture that equality holds.

4.7.2 Steiner $(5, 3)$ trades

If T is a non-void Steiner $(5, 3)$ trade of volume m, then Theorem 4.42 gives a lower bound of eleven for m. Now Theorem 4.42 is proved by counting $(t-1)$-subsets. We can obtain a stronger result by counting the multiplicity of subsets of $F(T)$ of other sizes.

Lemma 4.44: *Suppose that $T = T_1 - T_2$ is a non-void Steiner $(5, 3)$ trade of volume m. Then $m > 14$.*

Proof: Let $B = 12345$ be an arbitrary block in T_1. As in the proof of Theorem 4.42, the ten pairs in B must occur again, and in separate sets of T_1. Each point of B

has now occurred five times in T_1. Now, for any $x \in F(T)$, the restriction $R_x(T_1)$ is a $(4,2)$ trade, and so cannot have volume five. So the five points of B must have multiplicity at least six, and must occur at least once more in T_1. Since T is Steiner, they cannot occur with any of the ten pairs, nor can more than two of them occur in any block. So $m \geq 1 + 10 + 3 = 14$.

Now suppose that $m = 14$, and consider the partial structure for T_1 just discussed. The partial block in T_1 which contains a single element from B can contain, at most, one other point from B. The points from B cannot occur again in any of the other sets. So any element in $F(T)$ has multiplicity six or seven, and in any block from T_1 at most one element can have multiplicity seven.

Let n_6 (resp. n_7) denote the number of elements of $F(T)$ with multiplicity six (resp. seven); then, counting the total number of elements in T_1, $6n_6 + 7n_7 = 70$. This has only two solutions, $(n_6, n_7) = (0, 10), (7, 4)$. However, if $n_7 > 2$, there is a block of T_1 that contains at least two multiplicity seven points, a contradiction. □

A solely 2-balanced family with block size four can be constructed by extending the difference set method used to construct cyclic designs. Consider the unique 2-$(13, 4, 1)$ design obtained from the projective plane of order three. Two distinct (but isomorphic) copies of this design can be obtained by developing the blocks 0146 and 0256, using modulus thirteen. It is easy to check that these two designs are solely 2-balanced. By developing these blocks to any larger modulus, we obtain the following result.

Lemma 4.45: *Let A_1 (resp. A_2) be the set of r blocks obtained by developing the block 0146 (resp. 0256) to the modulus r, $r \geq 13$. Then $A_1 - A_2$ is a Steiner $(4,2)$ trade of volume r. Further, A_1 and A_2 are solely 2-balanced.*

Proof: Recall the notation $\triangle B$. Now $\triangle\{0,1,4,6\} = \triangle\{0,2,5,6\} = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\}$, and $r \geq 2 \cdot 6 + 1 = 13$, so $A_1 - A_2$ is obviously a Steiner $(4,2)$ trade of the stated volume. Now consider triples from the starter blocks, with the points ordered by value; each of these gives rise to an ordered pair of differences. For example, the triple $(0, 4, 6)$ yields the pair $(+4, +2)$. Noting that $(+a, +b)$ and $(+b, +a)$ are not equivalent, it is easy to check that all eight triples are different. So no triple is repeated, and A_1 and A_2 are solely 2-balanced. □

We can use Lemma 4.8 with these families to construct Steiner $(5,3)$ trades, giving $\mathcal{S}_1(5,3) \supseteq \{26, 28, 30, \ldots\}$. Using Lemma 4.3 with these values now gives $\mathcal{S}_1(5,3) \supseteq$

$\{51, 53, 55, \ldots\}$.

Various searches for Steiner $(5, 3)$ trades, either by considering the difference between 'random' 3-$(v, 5, 1)$ designs (see Section 7.5) or by adding basic trades (see Section 3.6) produced many trades different from those constructed above. However, the only new volume obtained was 45. In fact, any permutation, such as (012), that shifts precisely three points of a 3-$(17, 5, 1)$ design – say, the inversive plane I of Figure 11.2 – generates a pair of designs whose difference is a Steiner $(5, 3)$ trade of volume 45 and foundation 17.

4.8 Conclusions

For $\mathcal{S}_1(k, 2)$, $k \geq 7$, our results leave undetermined the inclusion of odd volumes in the range $2k+3$ and $3k-4$. Very recently, it has been proved that such trades do not exist, so $\mathcal{S}_1(k, 2)$ is now fully determined. This proof, which builds on our results and uses essentially the same techniques, has been accepted for publication [72]. The material in Appendix D demonstrates that our techniques are robust enough to be applied to the more general problem of G-trades.

For $t > 2$, the $k = t+1$ case is special, since we can induct on t. As Subsection 4.7.1 illustrates, $s_i \in \mathcal{S}_1(t+1, t)$ for $0 \leq i \leq t$, and $m \in \mathcal{S}_1(t+1, t)$ for all large enough m. Proving the remaining values becomes increasingly difficult for higher values of t. Based on our results, we conjecture that $\mathcal{S}_1(t+1, t) = \mathcal{S}(t)$.

For $t > 2$ and $k > t+1$, the Steiner trade problem seems to be very difficult. The various theoretical and computational methods discussed do not produce Steiner trades with 'small' volumes for these values of t and k. Our results indicate that a lower bound stronger than that of Theorem 4.42 should be sought, but it is unclear what form this should take.

Chapter 5

Trades and defining sets in designs

The two themes of our thesis are trades and defining sets. Recall that any defining set of a design D must have at least one block in common with any trade in D. In this chapter we present a potpourri of results on defining sets which make use of this connection. Various bounds on $|d_s D|$ were reviewed in Chapter 2, both for the general case and for infinite families. In Sections 5.1 and 5.2 we present some further bounds. In Section 5.3 we consider the problem of defining sets in 1-designs. In the final section we address the question of how many minimal trades or defining sets a design may contain.

The material presented in Section 5.1 has been published in [101], and has recently been extended by other workers [34]. The bound given in Section 5.2 has been published as part of [100].

5.1 An infinite family

Recall that $\mu = |d_s D|/b$. Putting $t = 2$ and $k = 3$ in Theorem 2.20, and noting that $b = v(v-1)/6$ in a $STS(v)$, we have that $\mu \geq 3/v$ for a $STS(v)$. Note that $\lim_{v\to\infty} 3/v = 0$. We describe an infinite family of $STS(v)$ where at least a quarter of the blocks are necessary to form a defining set.

The following consequence of Lemma 1.8 is trivial, following directly from the disjointness of the trades. However, it seems not to have been recorded in the literature.

Corollary 5.1: *Let $D = (V, \mathcal{B})$ be a t-(v, k, λ) design, and suppose that $\mathcal{B}$ contains m mutually disjoint (v, k, t) trades. Then $|d_s D| \geq m$.* □

A number of authors have examined the problem of partitioning the family of blocks of a Steiner triple system into sets of blocks isomorphic to a given set of blocks; such a partitioning is called a **decomposition**, and the given set of blocks is called a **configuration**. See, for example, [55, 54, 66]. The configuration that interests us is the Pasch configuration. Recall that this is the unique smallest $(3, 2)$ trade. In [1], the following result is proved.

THEOREM 5.2: *For all* $v \equiv 1, 9 \pmod{24}$, $v \geq 25$, *there exists a* $STS(v)$ *which can be decomposed into Pasch configurations.* □

Combining Corollary 5.1 and Theorem 5.2 immediately gives our required family.

THEOREM 5.3: *For all* $v \equiv 1, 9 \pmod{24}$, $v \geq 25$, *there exists a* $STS(v)$ *with* $\mu = |d_sD|/b \geq 1/4$. □

REMARK: The example Pasch-decomposable $STS(25)$ given in [1] is cyclic, with the four starter blocks forming the Pasch configuration {016, 02a, 1ad, 26d}. Attempts to find $|d_sD|$ for this design using the computational techniques described later were not successful, due to the large number of blocks. However, a defining set of 47 blocks was found, so $0.25 \leq \mu \leq 0.47$.

5.2 An upper bound

In several places throughout this thesis, we use the smallest volume trades in a design to give an upper bound on $|d_sD|$, a bound on $\max(\text{spec}_m(D))$, or an upper bound on the size of a partial that is not a defining set.

Since non-void (k, t) trades have volume at least 2^t, any set of at least $b - 2^t + 1$ blocks from a design must be a defining set. However, a t-(v, k, λ) design need not contain a trade of volume 2^t; for example, the unique $STS(9)$ does not contain any Pasch configurations. If we consider the smallest volume trades in a design, then we are led, via Lemma 1.9, to the following result.

LEMMA 5.4: *Let* $D = (V, \mathcal{B})$ *be a* t-(v, k, λ) *design with* $|\mathcal{B}| = b$, *and let* m *denote the size of a smallest volume non-void trade in* D. *Then any collection,* $\mathcal{C} \subseteq \mathcal{B}$, *of more than* $b - m$ *blocks from* D *completes uniquely.*

PROOF: If $\mathcal{C} = \mathcal{B}$, the result is trivial. Let $\mathcal{C}$ be a collection of blocks from $\mathcal{B}$, with $b - m < |\mathcal{C}| < b$, and suppose that $\mathcal{C}$ completes to two distinct designs D_1 and D_2. Now $\mathcal{B}_1 \backslash \mathcal{C} \neq \mathcal{B}_2 \backslash \mathcal{C}$, and both $\mathcal{B}_1 \backslash \mathcal{C}$ and $\mathcal{B}_2 \backslash \mathcal{C}$ are non-empty. Thus $(\mathcal{B}_1 \backslash \mathcal{C}) - (\mathcal{B}_2 \backslash \mathcal{C})$ contains a trade with volume less than m, which is not possible. □

COROLLARY 5.5: *For any minimal defining set,* $|d_sD| \leq |d_mD| \leq b - m + 1$. □

5.3 Defining sets for 1-designs

The sizes of smallest defining sets of 1-designs have not been considered in the literature. However, these are of interest; in particular, given the relationship between the sizes of smallest defining sets of a design and its extensions, they can assist in finding $|d_sD|$ for designs with $t > 1$. For 1-designs in general, we expect that $|d_sD| = b - x$, where x is ‘small’, since 1-designs contain large numbers of small-volume trades. However, there are non-trivial cases where $b - |d_sD|$ is arbitrarily large.

Lemma 5.6: (1) *For any* $x > 0$ *there exists a 1-design* D *with* $b - |d_sD| \geq x$*;*
(2) *There exists a 1-design with* $\mu \leq 1/2$*.*

Proof: To prove both parts, consider the 1-$(4, 2, x)$ design formed by taking x copies of $\{12, 34\}$; this has $2x$ blocks. Obviously, the partial consisting of x copies of the block 12 completes uniquely, and is a smallest defining set. □

To obtain more precise results, we consider when a collection of blocks can be, or can contain, a 1-trade. For $(k, 1)$ trades of volume two, it is simple to see that any pair of blocks A and B have a trade mate if and only if $|A \cap B| \leq k - 2$.

Definition 5.7: *A design* $D = (V, \mathcal{B})$ *is said to be* ***i-simple*** *if* $|A \cap B| \leq k - i$ *for all* $A, B \in \mathcal{B},\ A \neq B$*.*

Lemma 5.8: *Let* $D = (V, \mathcal{B})$ *be a* 1-(v, k, λ) *design. Then* $|d_sD| \leq b - 1$*, with equality if and only if* D *is* 2*-simple.*

Proof: That $|d_sD| \leq b - 1$ is obvious. If D is 2-simple, then any two blocks of D are a $(k, 1)$ trade, so $|d_sD| = b - 1$. If $|d_sD| = b - 1$, then any two blocks of D are a $(k, 1)$ trade, since if not there would be a partial of $b - 2$ blocks which completed uniquely. So any two blocks intersect in at most $k - 2$ points, and D is 2-simple. □

Corollary 5.9: *Let* $D = (V, \mathcal{B})$ *be a* t-$(v, k, 1)$ *design. Then, considered as a 1-design,* $|d_sD| = b - 1$*.*

Proof: Since D is Steiner, and $k \geq t + 1$, then D is 2-simple. □

We now consider a sufficient condition for a set of blocks not to contain a 1-trade.

Definition 5.10: *Let* $D = (V, \mathcal{B})$ *be a* t-(v, k, λ) *design, with* $|\mathcal{B}| = b$*. Let* $\binom{V}{l}$

denote the set of all l-subsets of V. Define

$$s(D) = \max_{S \in \binom{V}{k-1}} \left(|\{B : S \subseteq B \in \mathcal{B}\}| \right).$$

If S is any set of $s(D)$ blocks satisfying this definition, then any collection of blocks from S intersect in at least $k-1$ points, and thus cannot have a trade mate. Note however, that if S is any set of blocks any pair of which intersect in at least $k-1$ points, then it is not necessarily the case that the blocks of S have a $(k-1)$-subset in common – consider the collection {12,13,23}.

THEOREM 5.11: *Let $D = (V, \mathcal{B})$ be a 1-(v, k, λ) design, with $|\mathcal{B}| = b$. Then $|d_s D| \leq b - s(D)$.*

PROOF: Let S be a set of $s(D)$ blocks having a $(k-1)$-subset in common. Obviously S does not contain any 1-trade, and so $\mathcal{B} \setminus S$ completes uniquely. Thus $|d_s D| \leq b - s(D)$. □

5.4 Number of trades and defining sets

A collection of subsets is called an **antichain** (or *Sperner family*) if no member of the collection properly contains another. The collections of minimal trades and of minimal defining sets in a simple design are obviously antichains. The following two results are standard; see, for example, [3].

THEOREM 5.12: *Let $\mathcal{A}$ be an antichain of subsets of an n-set. Then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor},$$

with equality if and only if $\mathcal{A}$ consists of all the $\frac{n}{2}$-subsets if n is even, or all the $\frac{n-1}{2}$-subsets or all the $\frac{n+1}{2}$-subsets if n is odd. □

THEOREM 5.13: (The LYM Inequality) *Let $\mathcal{A}$ be an antichain of subsets of an n-set, and suppose that p_i members of $\mathcal{A}$ have cardinality i, for $0 \leq i \leq n$. Then*

$$\sum_{i=0}^{n} \frac{p_i}{\binom{n}{i}} \leq 1.$$ □

These two results give bounds on the number of minimal trades or minimal defining sets in a simple design, with the LYM Inequality being the stronger of the two. We call the sum in Theorem 5.13 the **w-size** (for *weighted size*) of the antichain, and use w_d (resp. w_t) for the w-size of the antichain of minimal defining sets (resp. trades). In Chapters 7 and 11 we count the number of minimal trades and minimal

Table 5.1: Some example w-sizes

design	w_d	w_t
2-(6,3,2)	0.250	0.048
2-(7,3,1)	0.800	0.200
2-(8,4,3), α^*	-	0.023
2-(8,4,3), β^*	-	0.032
2-(8,4,3), γ^*	-	0.056
2-(8,4,3), δ^*	-	0.033
2-(9,3,1)	0.382	0.039
2-(10,4,2), H_1	-	0.033
2-(10,4,2), H_2	-	0.015
2-(10,4,2), H_3	-	0.034
2-(13,3,1), D_1	0.019	-
2-(13,3,1), D_2	0.031	-
2-(13,4,1)	0.682	-
2-(16,4,1)	0.299	-
2-(21,5,1)	0.424	-
2-(25,5,1)	$\approx$ 0.15	-
2-(31,6,1)	$\approx$ 0.16	-
3-(8,4,1)	0.615	0.00233
3-(10,4,1)	? 0.255	0.000061
3-(17,5,1)	$\approx$ 0.13	-
4-(11,5,1)	$>$ 0.181	0.000017
5-(12,6,1)	$\geq$ 0.168	-

defining sets in a design. These results are collected and displayed in Table 5.1; a '$\approx$' indicates that the value is an estimate obtained by sampling, a '?' indicates that the value is correct if $\mathrm{spec}_m(D)$ has no holes, and a '$\geq$' or '$>$' indicates that only a partial enumeration was performed.

We do not expect the bound from the LYM Inequality to be tight, and this is borne out by our results. (We do not include the void trade in our count of trades, and we ignore the trivial cases where $|d_sD| = 0$.) However, for some designs the w-size is quite large, with w_d being more than one half in several cases.

In all the examples considered $w_d > w_t$, and the ratio w_d/w_t can be very large. For a given family of designs the w-sizes decrease as the designs get 'larger'; for example, w_d for the projective and affine planes decreases as the order increases. However, the w-sizes need not tend to 0 in a series of designs. In particular, for non-simple designs, although the number of different trades can be small, the number of times each is contained in the design can grow rapidly.

Example 5.14: Let D be a simple 2-$(v, 3, \lambda)$ design with b blocks which contains n copies of the Pasch trade. Now consider the 2-$(v, 3, x\lambda)$ design D' formed by taking

x copies of D. Each Pasch trade in D gives rise to x^4 Pasch trades in D', since there are x copies of each block. So

$$w_t \geq \frac{nx^4}{\binom{xb}{4}} = \frac{24nx^4}{xb(xb-1)(xb-2)(xb-3)} > \frac{24n}{b^4}. \qquad \square$$

REMARK: Any superset of a defining set is also a defining set, so the collection of all defining sets in a design forms an *upset* or *filter* [3], with the antichain of minimal defining sets being the minimal members of this upset. The collection of all trades in a design is not an upset.

Chapter 6

Trade enumeration in simple designs

Although any (v,k,t) trade is contained in some design (simply take the appropriate multiple of the full design), it is not necessarily the case that a particular design contains a particular trade. For a (k,t) trade T to be contained in a t-(v,k,λ) design it is obviously necessary that $v \geq f(T)$, $b \geq m(T)$, and that λ is 'large enough.' However, these conditions are not sufficient. For example, for all $v \equiv 3 \pmod 6$ there is an **anti-Pasch** $STS(v)$; that is, an $STS(v)$ with no trade of volume 4 [10].

To assist in investigating the trades contained in a t-(v,k,λ) design, an algorithm to enumerate all the trades in the simple designs with a given set of parameters was developed. We show how the lists of trades generated by the algorithm can be used to find the sizes of smallest defining sets of designs. By considering only some of the trades in a design, the technique discussed leads to the new concepts of *member* and *class* defining sets of designs.

Some of the material from this chapter and the next chapter – where the main results obtained using this algorithm are presented – has been published in [100].

6.1 Algorithm

Recall that n is the number of different designs with the same parameters, and suppose that all n of these designs are simple. Given the transversal $\mathcal{D} = \{D_0, \ldots, D_{n-1}\}$ we wish to enumerate, for each D_i, the subsets of D_i that are trades.

Suppose that $T_1 - T_2$ is a trade and that $T_1 \subseteq \mathcal{B}_i$ is a trade in D_i. Then, by the definition of a trade, $(\mathcal{B}_i \setminus T_1) \cup T_2$ is also a t-(v,k,λ) design, say D'. Now D' will be isomorphic to D_j, for some j. It may be that $i = j$, but this is not true in general. Note that, since T_1 and T_2 are non-empty and disjoint, $D' \neq D_i$. We say that T_1 is a trade **from** D_i **to** D' and that T_2 is a trade **from** D' **to** D_i.

Conversely, given D_i and any design $D' \neq D_i$, suppose that $\mathcal{B}_i = B \cup T_1$ and $\mathcal{B}' = B \cup T_2$, with $T_1 \cap T_2 = \emptyset$. Thus B is the set of blocks common to both designs,

FIGURE 6.1: The algorithm to enumerate all trades

```
1   for all permutations of $V$
2       for all $j$ in $0 \dots n-1$
3           permute design $D_j$
4           for all $i$ in $0 \dots n-1$
5               find the trade from $D_i$ to permuted $D_j$
6               process/store trade
7           end for
8       end for
9   end for
```

while $T_1, T_2 \neq \emptyset$, since $D' \neq D_i$. Now $T_1 - T_2$ is a trade, and D_i contains T_1. Note that, if $\mathcal{B}_i \cap \mathcal{B}' = \emptyset$, B will be empty, and the trade consists of the designs D_i and D' themselves.

So we can generate all the trades in a design D_i by comparing D_i to every other design with the same parameters and eliminating common blocks. Given $\mathcal{D}$, it is easy to generate all possible designs by applying all permutations of V to the designs in $\mathcal{D}$. Thus we arrive at the simple algorithm given in Figure 6.1.

This algorithm is obviously non-polynomial, since the outer loop 1–9 will be executed $v!$ times. It is possible to enumerate all permutations so that successive permutations differ by a single transposition, see for example [97, 103]. Such an enumeration yields a *constant amortised time* (CAT) algorithm, and thus the overhead to generate all the permutations of V is $O(v!)$.

For each iteration of the loop 1–9, statement 3 will be executed n times and statements 5 and 6 will be executed n^2 times each. If we restrict v to be less than the machine's word size (normally 32 bits) and implement sets as unsigned integers, then set operations are constant time, so the complexity of statement 3 depends only on the number of blocks in the design and will be $O(b)$. For statement 5, we have to compare every block in D_i with every block in the permuted D_j and strike out matched pairs. Since the permuted design is not sorted, this will be $O(b^2)$.

So, the overall complexity of the algorithm will be $O(v! + v!n(b + n(b^2 + C_6)))$, where C_6 stands for the complexity of statement 6. The trades are stored in binary trees, one tree to each of the n^2 ordered pairs of designs, with the trades ordered first by volume and then lexicographically. We call the first design (D_i) in the ordered pair the **initial** design and the second (D_j) the **final** design. For an ordered pair of designs, we regard trades as distinct if the sets of blocks of the trades in the initial design are different, and only these blocks are stored. We do not differentiate between trades with different sets of blocks in the final design; that is, we do not consider different trade mates.

Since comparing trades will be $O(b)$, searching the tree is an $O(b \log_2 N_T)$ operation, where N_T stands for the number of trades in the tree. Insertion of a new trade requires memory allocation and $O(b)$ time. We assume that the total time to execute statement 6 amortises to $O(b^2)$, or less, for each call. Thus, the complexity of the algorithm reduces to $O(v!n^2b^2)$. We investigate the validity of this expression empirically in Section 6.3. As some justification for our assumption, consider the following points:

- ▷ The trade may be void, in which case statement 6 can be skipped.
- ▷ The number of distinct trades is much less than the number of times, $v!n^2$, that statement 6 is executed. So, in most cases, only a partial traversal of the tree is required before the current trade is found and insertion proves unnecessary.
- ▷ The trees are kept as small as possible by there being one for each ordered pair of designs.
- ▷ The trees are initially empty, so the search time is initially small and grows as the tree grows.

Note that the running time of the algorithm is independent of the parameters t, k, λ, except as these affect b. In practice, calculating the permutations accounts for only a minor part of the running time of the algorithm. Profiled runs of the algorithm, using the `prof` Unix utility, revealed that the bulk of the time is spent building the trades in statement 5 or, to a lesser extent, comparing trades in the tree searches as part of statement 6.

The algorithm takes no account of the automorphism groups of the designs. Let A_i denote the order of the automorphism group of D_i and recall that N_i denotes the number of distinct designs isomorphic to D_i. Then $N = \sum_{i=0}^{n-1} v!/A_i$ is the total number of distinct designs, and the maximum number of trades from D_i is $N - 1$. The maximum number of trades from D_i to designs isomorphic to D_j, $i \neq j$, is N_j, and each of these will be enumerated at least A_j times. The maximum number of trades from D_i to isomorphic designs is $N_i - 1$ and, if the permutation is an automorphism, the trade will be void.

In general, since a trade may have many trade mates, these upper bounds on the number of distinct trade halves in a design are poor. So the algorithm may do much redundant work, generating empty trades or trades already listed. We make use of the empty trades, and keep a count of their number. Since void trades only arise when the permutation leaves the set of blocks unchanged, this count is, in fact, the order of the automorphism group of the design. Of course, this value is available from other sources, but its calculation here is convenient and provides a check on the operation of the algorithm.

6.2 Defining sets

Given a design D, suppose that we have enumerated the family of distinct trades $\mathcal{T} = \{T^i\}_{i \in I}$ in D using our algorithm, and that $|\mathcal{T}| = d$. This family can be represented by a $d \times b$ incidence matrix $M = [m_{ij}]$, with $m_{ij} = 1$ if trade T^i contains block b_j and 0 otherwise. In a similar manner to that of [71], each row of M can be thought of as a linear inequality

$$m_{i,0}b_0 + \cdots + m_{i,b-1}b_{b-1} \geq 1.$$

Here the binary variables b_i, $0 \leq i \leq b-1$, are our unknowns, and they stand for the blocks of D. In view of Lemma 1.9, since $\mathcal{T}$ contains all trades in D, any solution to this system of d inequalities is a defining set for D. Further, in view of Lemma 1.8, any defining set for D will be a solution to the system of inequalities.

So we formulate the binary integer linear programme (BILP): minimise

$$\sum_{i=0}^{b-1} b_i \,,$$

subject to the system of inequalites represented by M, with $b_i \in \{0, 1\}$, $0 \leq i \leq b-1$. Any optimal solution to this BILP yields a smallest defining set, and thus the value of $|d_s D|$.

6.2.1 Minimising the collection of trades

As a practical matter, to reduce the number of inequalities, we may choose to minimise $\mathcal{T}$ before solving the system, since this does not affect the validity of the solutions. Our enumeration algorithm stores the family of trades on binary trees, ordered first by volume and then lexicographically. An in-order traversal of such a tree processes the trades in order of increasing volume. As each trade is processed it is compared with the previously processed trades having smaller volume. If it is a superset of such a trade, it is non-minimal. If not, it is minimal. It is never the case that a previously processed trade is a superset of a later trade. This naïve algorithm is quadratic in the number of trades, and suffices for our purposes. Merging trees or lists, if required, is straightforward.

REMARK: Note that building a separate list of minimal trades by copying across the data required is not necessary. The list can be built by 'threading' through the existing tree, at the small expense of an extra pointer field in each node.

6.2.2 Member and class defining sets

To motivate what follows, consider the problem of identifying a unique design D among the N t-(v, k, λ) designs. To do so, we need to supply information about D. If the information consists solely of blocks of D then it is a defining set. If the information also includes block intersection numbers then it is a *specifying set*, see [94]. An arbitrary collection of information about a design, sufficient to identify it uniquely, will be called an **establishing set**.

If an establishing set includes information about D that is invariant under isomorphisms, then we may be able to partition $\mathcal{D}$ into two or more parts and say which part D lies in. The invariants may or may not be sufficient to identify uniquely the isomorphism class to which D belongs. Examples of design invariants include whether or not the design is simple and the order of the automorphism group of the design. An example of establishing information about D that is not an invariant is the knowledge that the design does not contain a particular k-subset of V.

Given D, for each isomorphism class D_i^* our algorithm enumerates the trades in D to designs in D_i^*. Each such set of trades represents a BILP. An optimum solution of this BILP represents the smallest number of blocks of D required to ensure that no completion lies in D_i^*, or in $D^* \setminus \{D\}$ if $D_i^* = D^*$.

If $D_i^* = D^*$, then the trades in D are all to another member of the class D^*. Such trades will be called **member trades**, or **m-trades**. Any solution, not necessarily optimal, to the BILP formulated from the collection of all m-trades is a **member** defining set. A member defining set of D is denoted mD, and we note that, while it may have more than one completion, exactly one of these completions is in D^* and this completion is D. Just as we can define minimal and smallest defining sets, we can define minimal and smallest member defining sets. These are denoted m_mD and m_sD respectively.

If $D_i^* \neq D^*$, then the trades in D are to designs in another isomorphism class. Such trades will be called **class trades**, or **c-trades**. Any solution, not necessarily optimal, to the BILP formulated from the collection of all the c-trades in D is a **class** defining set. A class defining set of D is denoted cD, and we note that, while it may have more than one completion, all of these completions are in D^* and one of them is D. Just as we can define minimal and smallest defining sets, we can define minimal and smallest class defining sets. These are denoted c_mD and c_sD respectively.

A set of blocks that is a class defining set is a class defining set for every design to which it completes. It is possible for a set of blocks to be a member defining set for more than one design. For example, if a set of blocks S completes in only two ways, to two non-isomorphic designs D' and D'', then S is a member defining set of both D' and D''.

REMARK: Note that, unlike a member defining set, a class defining set is not an establishing set for a single design. It could, perhaps, be considered an establishing set for a class of designs.

Obviously, any defining set is also a member and a class defining set, and a member and a class defining set together constitute a defining set, so

$$|m_sD|,\ |c_sD| \ \leq\ |d_sD| \ \leq\ |m_sD| + |c_sD|.$$

In the case $n = 1$, $|m_sD| = |d_sD|$ and $|c_sD| = 0$. In the case $n > 1$, for each of these inequalities the results quoted in the next chapter contain examples where equality holds and where it does not, except that no non-trivial example is known where $|d_sD| = |m_sD| + |c_sD|$. (If D is the full design, then $|\mathrm{aut}(D)| = v!$ and so $|m_sD| = 0$ and $|c_sD| = |d_sD|$.) Our results also show that $|m_sD| < |c_sD|$, $|m_sD| = |c_sD|$ and $|m_sD| > |c_sD|$ are all possible.

Member defining sets are particularly interesting, since we can find $|m_sD|$ given only the design D, by using the list of m-trades in D generated by a simplified version of our algorithm. We do not need a transversal of all the n classes and, in fact, we need not know what n is. Since $|m_sD| = |d_sD|$ in at least two cases where $n > 1$, $|m_sD|$ is a potentially tight lower bound on $|d_sD|$. In general, $|c_sD|$ is a better lower bound for $|d_sD|$, being tight in many of the examples given, but it is not so readily calculable.

REMARK: If the proof of parts (1)–(3) of Theorem 2.20 is examined, we see that they go through when S is a member defining set, as opposed to a defining set [36]. Thus, if D is STF, we can use the lower bound of Theorem 2.20(3) for smallest member defining sets.

Suppose now that we wish to estimate the value of $|m_sD|$. That is, given a design $D \in D_i^*$, what is the smallest number of blocks of D that uniquely identifies it among all the designs in D_i^*? The total number of blocks in the N_i designs in D_i^* is bN_i, and the total number of k-subsets of V is $\binom{v}{k}$. So, each k-subset of V appears in an average of $bN_i/\binom{v}{k}$ designs of D_i^*. Let the factor $f = \binom{v}{k}/b$. Then $1/f$ is the average proportion of the designs in D_i^* that contain a given k-subset.

Now all designs in D_i^* are isomorphic, differing only in the labelling of the elements. If we assume that the k-subsets of V are randomly distributed among the N_i designs of D_i^*, then f is the reciprocal of the probability that a particular k-subset of V appears in a given design. So, the knowledge that a design contains a particular k-subset of V means that the design is one of N_i/f designs from the N_i designs in D_i^*. If we assume further that the blocks in a design are independent of each other, then the knowledge of x blocks of a design means that the design is one of N_i/f^x designs from the N_i designs in D_i^*. To specify D uniquely, this value must be at most 1; that is, $N_i \le f^x$. Taking logarithms to base f yields $x \ge \log_f N_i$. The value $\log_f N_i$ is thus the expected value of $|m_sD|$, under the assumptions stated. We investigate the validity of this expression in Section 7.3.

6.3 Timing information

In an effort to validate the expression for the complexity of the algorithm, and to establish whether or not the cost of processing the trees of trades did, in fact, amortise to no more than $O(b^2)$, timing data was recorded for runs of the programme. This data is presented in Table 6.1, and covers the parameter sets (for both $n > 1$ and $n = 1$) discussed in the next chapter. The first column gives the parameters of the design, with the next two columns giving n and b. The next two columns give the value of $v!n^2b^2$ and this value normalised to that for the 2-$(8, 4, 3)$ designs. The final two columns give the running times to generate all the trades in all the designs, both actual and normalised.

The running time is the amount of actual CPU time used by the programme, and does not include system or I/O time. The runs were performed on a Sun-4m SPARC-based server, with a 100MHz clock. Note that the time does not include the time

Table 6.1: The running times to generate all trades

design	n	b	$v!n^2b^2$	norm	time	norm
2-$(6,3,2)$	1	10	72000	0.00057	0m00.03s	0.00072
2-$(7,3,1)$	1	7	246960	0.00195	0m00.10s	0.00239
3-$(8,4,1)$	1	14	7902720	0.06250	0m01.73s	0.04138
2-$(9,3,1)$	1	12	52254720	0.41327	0m16.42s	0.39273
2-$(8,4,3)$	4	14	126443520	1.00000	0m41.81s	1.00000
3-$(10,4,1)$	1	30	3265920000	25.8291	11m28.90s	16.4769
2-$(11,5,2)$	1	11	4829932800	38.1983	21m42.70s	31.1576
2-$(10,4,2)$	3	15	7348320000	58.1154	33m46.09s	48.4595
2-$(9,4,3)$	11	18	14226347520	112.511	1h44m01.72s	149.288
4-$(11,5,1)$	1	66	173877580800	1375.14	7h56m14.11s	683.428
4-$(11,6,3)$	1	66	173877580800	1375.14	8h05m24.27s	696.586
3-$(10,5,3)$	7	36	230443315200	1822.50	14h26m53.62s	1244.05
2-$(10,5,4)$	21	18	518497459200	4100.63	64h10m47.71s	5526.14

to process the list of trades – say to extract and minimise a particular collection of trades – nor the time to solve any BILP problem. These additional times can be non-trivial.

The normalised running times are in good agreement with the times obtained using the expression for the complexity, with the ratio of actual to 'predicted' time varying from 1/2 to 4/3. So it appears that tree processing times can be ignored in assessing the complexity of the algorithm, and that $v!n^2b^2$, suitably normalised, predicts the running time to within a factor of two.

6.4 Variations of the algorithm

Despite its simplicity, the algorithm presented has proved effective, as evidenced by the results given in the next chapter. However, due to its high complexity, extending its reach to other parameter sets would require substantial efficiency improvements. One possible approach would be to take into account the automorphism groups of the designs. Consideration of only the distinct designs in each class would replace the $n^2v!$ term in the expression for the complexity by the term $n\sum_{i=0}^{n-1} v!/A_i = nN$. This represents a reduction in the amount of work by a factor of $n/(\sum_{i=0}^{n-1} 1/A_i)$.

This is a potentially significant reduction, with the actual value depending on the automorphism group orders. Parameter sets where one or more of the designs has a trivial automorphism group yield smaller reductions. For the five parameter sets with $n > 1$ given in Table 6.1, the work would be reduced by factors of 26.2, 47.0, 3.0, 19.1 and 2.7. However, to realise this reduction we would have to generate, for each design, a set of coset representatives of the automorphism group in the group of permutations of V. Whether or not the complexity of doing this would be outweighed by the reduction in the number of executions of statements 5 and 6 of the algorithm has not been investigated.

In cases where the running time is too long, a modified version of the algorithm can be used to generate partial lists of trades. This can be done in several ways: by imposing a time limit; by imposing a limit on the number of trades generated; or by generating only some of the permutations. The first two of these are straightforward and effective, since a significant proportion of the distinct trades are found early in a run. The last technique is the most versatile, although its efficacy depends heavily on the particular designs and the trades 'required.' Three possible implementations of this technique are: generate the first permutations in some order; generate some number of random permutations; or generate only those permutations that are *derangements* of exactly u of the v points, for some $1 < u \leq v$.

Partial lists of trades can be used to find lower bounds for $|m_sD|$, $|c_sD|$ and $|d_sD|$, in the manner discussed in [71], to help eliminate sets of blocks from consideration as defining sets, as discussed in [17, 53, 51], or as part of an investigation of trades, as discussed in Section 7.5.

As our results indicate, the number of trades can be very large, and this can cause memory or disk-space problems. To overcome these it may be necessary to modify the algorithm to generate only those trades required, or to process the trades as they are generated. The algorithm only stores trade halves, not full trades. This is done partly to reduce the size of the data structures needed to store the trades, and partly because, since our intended application is finding the size of defining sets, we do not need the discarded information. Additionally, distinguishing trades on the basis only of the sets of blocks in the initial design cuts down the number of 'distinct' trades generated. If we were interested in studying the number or structures of trade mates, it would be straightforward to amend our algorithm to record these.

If it is not possible to generate all the trades in all the designs at once, it is simple to modify the algorithm to generate and store only the trades in a particular design. Thus n runs, each taking $O(v!nb^2)$ time, are required to generate all the trades in all the designs, but the memory usage for each run is decreased by a factor of approximately n.

6.5 Conclusions

The algorithm described has proved very useful, and we make free use of trades produced by it, without commenting on their provenance. The specific purpose for which the algorithm was originally written – that is, defining sets – is discussed in the next chapter, along with some details of the trades in the designs considered.

One area not addressed by our algorithm is that of non-simple designs. Here the design, and their trades, are not sets, but multisets. The algorithm presented generates a representative of each trade which uses blocks repeated in the design, but does not generate a full list of these trades, since we need to distinguish between 'identical' blocks. When the trade contains blocks which are also contained in the untraded portion of the design, the duplicated blocks of the trade cannot be used to distinguish between the initial and the final designs. The *discriminating sets* described in Chapter 8 are intended to address these problems.

REMARK: Given a set of parameters with $n > 1$ it is often the case that some of the designs are simple and some are non-simple. The algorithm generates a full list of trades in the simple designs, enabling $|m_sD|$, $|c_sD|$ and $|d_sD|$ to be calculated for these designs. For the non-simple designs, the partial list of trades generated could be used to find lower bounds for these values.

Chapter 7

Results on simple designs

In this chapter we present some results obtained by enumerating trades in designs. In Sections 7.1 and 7.2 we use the algorithm of the previous chapter to enumerate all trades in simple designs. We give counts of the number of various trades in the designs and calculate the sizes of smallest member, smallest class, and smallest defining sets. In Section 7.3 we discuss the accuracy of our estimate for $|m_s D|$. In Section 7.4 we discuss the distribution, by volume, of the trades in the designs. Finally, in Section 7.5 we indicate how even a partial enumeration of trades can assist in their study.

7.1 Trade enumerations and defining sets

In this section we present results for five sets of parameters, where $n > 1$ and all the designs are simple. Each set of results is briefly discussed in a subsection. The results themselves are presented in tabular form in Appendix E, with three tables, or pairs of tables, per parameter set.

The first table for each set lists the number of distinct sets of tradeable blocks in each design. The first column of this table gives the label of the design, as given in the reference from which the transversal is drawn. The next n columns list the number of distinct trades from each design to each of the other designs; that is, to designs in the given isomorphism class. The number of m-trades in D_i can be obtained from column D_i of this $n \times n$ array of values. The c-trades column lists the total number of distinct c-trades in the design. The final column lists the total number of distinct trades in the design.

Note that the number of trades listed in these last two columns can be less than the sum of the number of trades in the appropriate columns from the first n. This is because a trade in a design may trade to both isomorphic and non-isomorphic designs, and may have many or few trade mates. The variation in the number of trade mates also explains why the $n \times n$ array of values is not symmetric; that is, the number of trades from D_i to D_j need not equal that from D_j to D_i.

The number of distinct c-trades in D_i is bounded below by the maximum number of c-trades from D_i to each of the non-isomorphic designs. The total number of distinct trades in D_i is bounded below by the maximum of the number of m-trades and the number of c-trades. The tables contain examples where the number of c-trades and trades are equal. However, there are no examples where these numbers match those for a particular initial/final pair of designs.

The second table lists the sizes of the collections of trades in the same manner as the first, but here the collections of trades have been minimised. That is, any trade which is a proper superset of another trade in the collection has been removed. Note the significant, but very variable, reduction in the number of trades after minimisation.

The final table for each parameter set gives the order of the automorphism group, A_i, and the number of distinct designs, $N_i = v!/A_i$, in each isomorphism class. The logarithm of N_i, to the base f, gives the expected value of $|m_sD|$, under the assumption that the blocks in a design are random and independent.

The values of $|m_sD|$, $|c_sD|$ and $|d_sD|$, found by solving the BILP optimisation problems represented by the lists of appropriate trades, are given in the final three columns. The values of $|m_sD|$ and $|c_sD|$ are all new. The values of $|d_sD|$ for four of the parameter sets have previously been calculated. Our results match the published results, except in one case, which is discussed in the relevant subsection. The values of $|d_sD|$ for the 21 2-$(10, 5, 4)$ designs are new, although they were obtained independently at the same time in [74]; our results were reported in [100].

Our rationale for including this wealth of material regarding trades between pairs of designs is to highlight the very variable nature of the results. In the subsections that follow, we remark upon various points of interest; however we are unable to draw any general conclusions from the data. It does, however, provide a ready source of counterexamples for refuting conjectures.

7.1.1 The 2-$(8, 4, 3)$ designs

There are four non-isomorphic 2-$(8, 4, 3)$ designs. The transversal used here is that given in [47], which also gives $|d_sD|$ for each design.

Note that there are 30 distinct designs isomorphic to γ^* and that there are 30 distinct trades from δ^* to designs isomorphic to γ^*. So the upper bound on the number of distinct trades given in Section 6.1 is attained, and all the designs isomorphic to γ^* can be generated from δ^* by trading different sets of blocks of δ^*.

7.1.2 The 2-$(10, 4, 2)$ designs

There are three non-isomorphic 2-$(10, 4, 2)$ designs, with each design being a residual design of a 2-$(16, 6, 2)$ design. The transversal used here is that given in [52], which also gives $|d_s D|$, and enumerates all smallest defining sets. Note that $|m_s H_3| = |c_s H_3| = |d_s H_3| = 5$.

7.1.3 The 2-$(9, 4, 3)$ designs

There are 11 non-isomorphic 2-$(9, 4, 3)$ designs. The transversal used here is that given in [95], which also gives $|d_s D|$, counts the number of distinct smallest defining sets, and lists several examples for each design.

Note that the transversal can be partitioned into the parts $\{\mathcal{M}_1, \mathcal{M}_2\}$, $\{\mathcal{M}_3, \mathcal{M}_4\}$, $\{\mathcal{M}_5, \mathcal{M}_6, \mathcal{M}_7\}$, $\{\mathcal{M}_8, \mathcal{M}_9\}$ and $\{\mathcal{M}_{10}, \mathcal{M}_{11}\}$. Within each of these parts, the total number of distinct, or minimal distinct, trades is the same. These parts match the possible extensions to 3-$(10, 5, 3)$ designs, see [95]. For example, a 2-$(9, 4, 3)$ design extends to the 3-$(10, 5, 3)$ designs $\mathcal{N}_1$ or $\mathcal{N}_2$ if and only if it is design $\mathcal{M}_1$ or $\mathcal{M}_2$. Interestingly, $\mathcal{M}_8$ and $\mathcal{M}_9$ have more than twice as many distinct minimal c-trades and trades as any of the other designs, but they have the lowest values of both $|c_s D|$ and $|d_s D|$.

7.1.4 The 3-$(10, 5, 3)$ designs

There are seven non-isomorphic 3-$(10, 5, 3)$ designs, all of which are extensions of 2-$(9, 4, 3)$ designs. The transversal used here is that given in [95], which also gives $|d_s D|$, counts the number of distinct smallest defining sets, and lists several examples for each design. Note that the total number of distinct minimal c-trades in the design $\mathcal{N}_4$ is less than the number of such trades from $\mathcal{N}_4$ to any of the individual non-isomorphic designs.

The value calculated for $|d_s D|$ for design $\mathcal{N}_1$ does not match the value of 6 originally given in [95]. The blocks of design $\mathcal{N}_1$, after sorting into lexicographic order, are:

01247, 01259, 01268, 01346, 01358, 01379, 01489, 01567, 02348,
02357, 02369, 02456, 02789, 03459, 03678, 04578, 04679, 05689,
12345, 12367, 12389, 12469, 12578, 13478, 13569, 14568, 14579,
16789, 23479, 23568, 24589, 24678, 25679, 34567, 34689, 35789.

The underlined values are an optimal solution of the BILP, in five blocks. That this putative defining set of five blocks from $\mathcal{N}_1$ completes uniquely was checked by performing partial completions by hand and then using the `complete` utility (see [18]) to find all completions.

First note that $b = 36$, $r = 18$, and that each pair of points occurs in $\lambda_2 = 8$ blocks. Consider the five underlined blocks. The triple 345 has not yet appeared, so the block 345-- must be in any completion three times. Each of the pairs 34, 35 and 45 has now appeared four times. Thus the blocks 34---, 35--- and 45--- must appear four more times each. The elements 3, 4 and 5 have now appeared 13, 14 and 13 times each respectively. Thus the blocks 3----, 4---- and 5---- must appear 5, 4 and 5 more times each respectively. This gives, in partial form, all but two of the blocks of the design.

The element 8 has appeared four times, so must appear in fourteen other blocks. Each of the triples 348, 358 and 458 has appeared once, so each must appear twice more. We distinguish three cases, with the block 3458- occuring zero, one or two times. Taking the value of λ_2 into account, we obtain the three respective partial completions:

02348, 04679, 16789, 23568, 24589, 345, 345, 345, 348, 348, 34, 34, 358, 358, 35, 35, 458, 458, 45, 45, 38, 38, 3, 3, 3, 48, 48, 4, 4, 58, 58, 5, 5, 5, 8, 8;

02348, 04679, 16789, 23568, 24589, 3458, 345, 345, 348, 34, 34, 34, 358, 35, 35, 35, 458, 45, 45, 45, 38, 38, 38, 3, 3, 48, 48, 48, 4, 58, 58, 58, 5, 5, 8, -;

02348, 04679, 16789, 23568, 24589, 3458, 3458, 345, 34, 34, 34, 34, 35, 35, 35, 35, 45, 45, 45, 45, 38, 38, 38, 38, 3, 48, 48, 48, 48, 58, 58, 58, 58, 5, -, -.

These three partial completions were used as input to the `complete` utility. The first of them completed uniquely, to $\mathcal{N}_1$. The other two have no completions to 3-$(10, 5, 3)$ designs. Thus, the set of blocks found by solving the BILP generated from the trades is a defining set, and $|d_s\mathcal{N}_1| = 5$. This being the case, the comment at the end of [95] regarding a case where the unique extension of a design D has a smaller smallest defining set than D is incorrect. (Note that the corrigendum to [95] was issued after its author was made aware of our results.)

Note that $\mathcal{N}_1$ has very many more minimal trades than any of the other designs, yet has the smallest value of $|d_sD|$. Similarly, $\mathcal{N}_5$ and $\mathcal{N}_6$ have more trades than the remaining four designs, yet smaller values of $|d_sD|$.

7.1.5 The 2-$(10, 5, 4)$ designs

The 21 non-isomorphic 2-$(10, 5, 4)$ designs have been enumerated in [29, 113]. Each of these designs is the residual of a 2-$(19, 9, 4)$ Hadamard design, see [94, 113]. The transversal used here is that given in [113], with the bracketed numbers in the first column of Table E.19 giving the numbering used in [29] – `nauty` was used to perform the cross-referencing.

The values of $|d_sD|$ for these designs have not previously been given. For convenience, the blocks of these designs are given in Tables E.17 and E.18, in lexicographic order. Sample smallest member defining sets, class defining sets and defining sets are marked in these tables. The defining sets are indicated by 'flagging' the blocks in them with a '-' symbol. The position of this symbol – bottom, middle, top – indicates which type of smallest defining set – resp. member defining set, class defining set, defining set – the block is in. These defining sets were those obtained from optimal solutions to the BILP. No attempt was made to find smallest member or class defining sets which were subsets of smallest defining sets, or were distinct from smallest defining sets.

Note that, in most cases, $|c_sD| = |d_sD|$ and that the smallest class defining set given is also a smallest defining set. However, this is not always the case. For example, for design D_5, $|c_sD_5| = |d_sD_5| = 6$ and the smallest class defining set found differs from the smallest defining set. Further, the smallest class defining set is a *proper* class defining set, in the sense that it has more than one completion. In fact it has two distinct, but isomorphic, completions. One of these is D_5, and the other is:

01269, 01357, 01456, 01789, 02348, 02358, 02479, 03469, 05678,
12359, 12367, 12478, 13468, 14589, 24567, 25689, 34579, 36789.

Note that as given in [113] the pairs of designs 1 and 2, 3 and 4, 6 and 7, 8 and 9, 10 and 11, 12 and 13, and 16 and 17 are complementary. Designs 20 and 21 are isomorphic to each other's complement. The remaining five designs are isomorphic to their own complements. The values of $|m_sD|$, $|c_sD|$ and $|d_sD|$ are the same for the complementary pairs, as are the total numbers of m-trades, c-trades and trades. This pairing of designs yields many 'symmetries' in the tables. For example, the

Table 7.1: Some simple designs, with $n = 1$

design	A_0	N_0	f	$\log_f N_0$	distinct	minimal	$\lvert d_sD\rvert$
2-$(6, 3, 2)$	60	12	2	3.58	11	10	3
2-$(7, 3, 1)$	168	30	5	2.11	15	7	3
2-$(9, 3, 1)$	432	840	7	3.46	188	36	4
2-$(11, 5, 2)$	660	60480	42	2.95	298	66	5
3-$(8, 4, 1)$	1344	30	5	2.11	15	7	3
3-$(10, 4, 1)$	1440	2520	7	4.02	1526	415	4
4-$(11, 5, 1)$	7920	5040	7	4.38	4181	3465	5
4-$(11, 6, 3)$	7920	5040	7	4.38	4181	3465	5

number of trades from D_1 to D_2 is the same as the number from D_2 to D_1.

Note that, when compared with the expected value $\log_{14} N_i$, the value of $|m_sD|$ is not monotonic – the values for designs D_{16} and D_{17} are too low.

7.2 Some $n = 1$ examples

Although the intended use of the algorithm is in the case where $n > 1$, it can be run where $n = 1$. We can obtain a count of the number of distinct and minimal distinct trades in these designs, all of these being m-trades. If $|d_sD|$ is not known, it can be calculated from these lists of trades. Since $|m_sD| = |d_sD|$, this provides further test data concerning our expression for the expected value of $|m_sD|$. Additionally, it provides a wide range of $v!n^2b^2$ values on which to perform timing tests.

Accordingly, in Table 7.1, we present the results of some runs in the $n = 1$ case, where the unique design is simple. The parameters of the design are listed in the first column, with the order of the automorphism group and the number of distinct designs listed in the following two columns. The next two columns contain the value of f and then the expected value of $|m_sD|$ (that is, $|d_sD|$). The next two columns list the number of distinct trades and the number of distinct minimal trades respectively. The final column lists the value of $|d_sD|$, obtained by solving the BILP optimisation problems represented by the trades. These values match those available in the literature [46, 47, 52].

Note that, for the 2-$(6, 3, 2)$ design, the number of distinct non-minimal trades is equal to the upper bound of $N_0 - 1 = N - 1 = 11$. Thus, any 2-$(6, 3, 2)$ design can be generated from a given design by trading a different set of blocks. The ten minimal trades are equally split between volume four and volume six trades, while the non-minimal trade is that to the disjoint design.

The 3-$(8,4,1)$ design is the unique extension by complementation of the 2-$(7,3,1)$ design and, apart from A_0, all the values in the table are the same for both designs. Note that the 3-$(8,4,1)$ design is also the 2-$(8,4,3)$ design γ^*. So the number of trades matches the number of m-trades, and $|d_sD|$ matches $|m_s\gamma^*|$. The 4-$(11,5,1)$ and 4-$(11,6,3)$ designs are complements of each other, and all the values in the table are the same for both designs.

7.3 Expected value of $|m_sD|$

To obtain an expression for the expected value of $|m_sD|$, we assumed that blocks in a design are independent. This assumption is obviously incorrect, given that the collection of blocks in a design is t-balanced. Despite this, the results quoted show that $\log_f N_i$ is a reasonable estimate of $|m_sD|$ for the simple designs considered, being within distance 1 in all but two cases where $n > 1$ and one case where $n = 1$. The estimate is neither consistently above nor consistently below the actual value, even within a set of designs with the same parameters. Unfortunately, $\log_f N_i$ is not monotonic with $|m_sD|$ – see the results for the 2-$(10,5,4)$ designs.

Note that, the more 'structure' a design has, the lower we would expect our prediction to be in relation to the actual value, since additional blocks in a defining set do not provide as much 'information' as the initial block. As an example, where $n = 1$ and $|m_sD| = |d_sD|$, consider the 2-$(11,5,2)$ design in Table 7.1. This design is linked, with a linkage of 2, and the actual value of $|m_sD|$ is greater than the expected value by more than 2.

7.4 The trade volumes

Our main purpose in enumerating the trades in a design is to use them to investigate defining sets. However, the lists of trades can yield other interesting information. As an example, in Tables E.20 and E.21 of Appendix E we provide some data on the distribution of distinct and minimal trades by volume and foundation. In these tables m and f stand respectively for the volume, $m(T)$, and the foundation, $f(T)$, of the trades. The number of distinct and minimal trades are given in the '#d' and '#m' columns respectively, with a '-' indicating that none of the distinct trades of this volume and foundation are minimal. The symbol '.' is purely a visual aid.

In Table E.20 note that the unique $SQS(8)$ is the unique extension by complementation of the unique $STS(7)$, and that the $SQS(8)$ is self-complementary. The basic trades in the two designs are related by Lemmas 4.43 and 4.8. The other trades in the $STS(7)$ are obtained as restrictions of the trades in the $SQS(8)$. Recall that γ^* is the 3-design considered as a 2-design. Each basic trade in the $SQS(8)$ yields eight basic trades in γ^*, as the leaves of the eight points. Each of the other trades in the $SQS(8)$ yields sixteen trades in γ^*, two for each of the eight points. The 2-$(8, 4, 3)$ designs have $b = 14$, and in three of the four designs any set of twelve blocks is a trade. So, for any pair of blocks in these designs, another design exists which is disjoint apart from the nominated pair of blocks.

In Table E.21 note that the 4-design is an extension of the 3-design, which, in turn, is an extension of the $STS(9)$. In these designs the $\binom{v}{2}$ trades of volume $2(r - \lambda_2)$ produced by a single-transposition permutation of V are always minimal, and are among the smallest volume trades in the designs. Considered as a 2-design, the $SQS(10)$ can be decomposed into two copies of H_1, see [52]. The 2-$(10, 4, 2)$ designs have $b = 15$, and in all four designs any set of twelve blocks is a trade. So, for any triple of blocks in any of these designs, another design exists which is disjoint apart from the nominated triple of blocks.

7.5 Searches for trades

When investigating the structure or spectrum of trades, one convenient source of examples is our enumeration algorithm, or its variants. It is a simple matter to print out any trades of interest, or to log the volume/foundation combinations found. Obviously, simple designs yield simple trades, and Steiner designs yield Steiner trades.

As an example, when investigating $\mathcal{S}_1(5, 3)$ the volumes of the trades in the 3-$(17, 5, 1)$ design were recorded, in an attempt to find Steiner $(5, 3)$ trades of 'small' volume. Only a partial enumeration can be performed using our algorithm, since $v!n^2b^2 = 17!1^268^2 \approx 1.6 \times 10^{18}$. This incomplete enumeration produced trades of volumes 30, 44, 45, 48, 51, 52 and 54, …, 68. A volume 45 Steiner $(5, 3)$ trade was previously unknown, and the one found is that given in Subsection 4.7.2.

This technique has been found to be very useful for producing large numbers of trades in a design, with many different volumes, foundations, and structures. Throughout our investigations, such incomplete enumerations were used to find example trades or to investigate the trades in a particular design.

7.6 Concluding Remark

When solving the BILP, no attempt was made to enumerate or analyse all optimal solutions, and thus all smallest defining sets. One interesting question for further investigation is: “under what circumstances can a smallest class defining set, or a smallest member defining set, be embedded in a smallest defining set?”

Chapter 8

Reduced discriminating sets

For simple designs, the problem of finding smallest defining sets can be addressed by enumerating trades and solving the BILP optimisation problems represented by these, as discussed in Chapter 6. For non-simple designs, Lemmas 1.8 and 1.9 are still valid, but repeated blocks introduce problems in the handling of trades.

Example 8.1: Consider the 2-$(7, 3, 2)$ design D consisting of two copies of the Fano plane of Example 2.1. This contains the Pasch trade $\{013, 124, 235, 450\}$. However, due to the repeated blocks, D contains sixteen 'copies' of this particular trade. □

We must distinguish between all these copies of a trade, and intersect them all in any defining set. So we need to uniquely label the blocks of a design, and consider the design as a multiset instead of a set. We could proceed using trades in this fashion, but we are not interested here in the trades per se, only in their ability to help us find defining sets. The difference between two designs is a trade, and we are actually using these trades to discriminate between designs. Developing this idea leads to the notion of a *reduced discriminating set* and we show how collections of these can be used to find defining sets for non-simple designs.

8.1 Multiset notation

Recall that $\mathcal{B}$ is the set of blocks of a design D, that $\mathcal{B}$ contains b blocks, and that b^* of these are distinct. For a non-simple design, $b^* < b$. We let $\mathcal{B} = \{b_0, \ldots, b_{b-1}\}$. As a matter of convention, and for later convenience, the blocks of D are assumed to be labelled in lexicographic order. Note that this means that repeated blocks will have consecutive subscripts.

We represent a multiset M of subsets of V as a set of ordered pairs (s_i, m_i), where s_i is a subset of V and m_i is its multiplicity in M. The ordered pairs will be called **melts** – for *multiset elements* – and two melts (s_i, m_i), (s_j, m_j) are equal if and only if $s_i = s_j$ and $m_i = m_j$. We require that $m_i > 0$ (that is, we do not record subsets of zero multiplicity) and that, if $s_i = s_j$, then $i = j$. The cardinality $|M|$ of M is the number of distinct subsets of V in M; that is, the support size of M. The total number of subsets of V in M is denoted by $T(M)$, with $T(M) = \sum_i m_i$.

When our multiset M is a design D, or a submultiset of a design, the s_i will be blocks of the design, drawn from $\mathcal{B}$. For definiteness, we will identify each s_i with the lowest indexed block b_j of $\mathcal{B}$ such that $b_j = s_i$. Thus, (s_i, m_i) stands for the blocks $b_j, \ldots, b_{j+m_i-1}$.

We define union ($\cup$), difference ($\setminus$) and intersection ($\cap$) for multisets in the following fashion. For union, we add the multiplicities of matching blocks thus, $\{(s, m_1)\} \cup \{(s, m_2)\} = \{(s, m_1+m_2)\}$. For difference, we remove a block's melt only if it has the same multiplicity in both multisets. That is, $\{(s, m_1)\} \setminus \{(s, m_2)\} = \emptyset$ if $m_1 = m_2$, and if $m_1 \neq m_2$ then the melt (s, m_1) is unchanged in the difference. This definition is, perhaps, not the most intuitive one, but it is the one that suits our needs. For intersection, we use the minimum of the multiplicities thus, $\{(s, m_1)\} \cap \{(s, m_2)\} = \{(s, \min\{m_1, m_2\})\}$.

We also define a binary operation $\sqcap$, analogous to intersection, but taking into account multiplicities. A melt (s, m) is in $M \sqcap M'$ if and only if $(s, m) \in M$ and $(s, m) \in M'$. Note that $D \setminus D' = D \setminus (D \sqcap D')$. We define the binary relation subset ($\subseteq$) in the obvious way; with $D \subseteq D'$ if and only if for every $(s, m) \in D$, $(s, m + e) \in D'$, for some $e \geq 0$. For convenience, given a melt (s, m), any melt $(s, m+e)$, $e > 0$, is called a **supermelt** of (s, m), and any melt $(s, m-e)$, $0 < e < m$, is called a **submelt** of (s, m).

REMARK: When a multiset is a set, then $\setminus$, $\cap$, and $\subseteq$ have their usual meanings, with $\cap$ and $\sqcap$ being equivalent. However, $\cup$ does not have its usual meaning.

8.2 Running example

To illustrate the ideas to be discussed, we will use the ten non-isomorphic 2-$(7, 3, 3)$ designs, as given in [47]. We label the indecomposable design I and the nine decomposable designs $R_1, \ldots, R_9$. Thus, our transversal $\mathcal{D} = \{I, R_1, \ldots, R_9\}$. The blocks of these designs, along with the automorphism group orders, A_i, the number of distinct designs in each isomorphism class, N_i, and the support sizes, b^*, are given in Table 8.1. The points have been relabelled $0 \ldots 6$ instead of $1 \ldots 7$, and the blocks have been arranged in lexicographic order; the starred entries will be discussed later.

Table 8.1: The ten 2-$(7,3,3)$ designs

design	I	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9
b_0	012	012	012	012	012	012	012	012	012	012
b_1	*014	012	012	012	012	012	012	012	012	012
b_2	016	012	012	*015	012	012	013	*013	013	013
b_3	024	034	034	026	*034	034	026	026	024	026
b_4	025	034	034	034	*034	035	034	034	034	034
b_5	034	034	035	034	035	*036	034	034	035	*035
b_6	035	056	046	035	046	045	045	045	046	045
b_7	036	056	*056	046	056	046	056	056	056	046
b_8	056	056	*056	056	056	056	056	056	056	056
b_9	123	*135	*134	123	135	134	124	125	*126	*124
b_{10}	125	*135	135	134	135	135	135	135	135	135
b_{11}	135	*135	135	135	136	136	135	136	136	136
b_{12}	136	146	146	146	145	145	*146	145	145	145
b_{13}	145	146	146	146	146	146	*146	146	145	146
b_{14}	146	146	156	156	146	156	156	146	146	156
b_{15}	234	236	236	236	234	234	235	234	234	234
b_{16}	236	236	236	236	236	235	236	235	235	235
b_{17}	246	236	236	245	236	236	236	236	236	236
b_{18}	256	245	245	245	245	*245	245	245	245	245
b_{19}	345	245	245	245	245	246	245	246	256	256
b_{20}	456	245	245	356	*256	256	*346	*356	346	346
A_i	42	168	24	8	12	144	21	3	6	6
N_i	120	30	210	630	420	35	240	1680	840	840
b^*	21	7	11	15	13	19	14	17	18	20

Example 8.2: As an example of the multiset representation used, consider $R_2 = \{(012,3), (034,2), (035,1), (046,1), (056,2), (134,1), (135,2), (146,2), (156,1), (236,3), (245,3)\}$. Here, $s_0 = b_0 = 012$, $s_1 = b_3 = 034$, $\ldots$, $s_{10} = b_{18} = 245$. Note that $|R_2| = 11 = b^*$ is the support size of the design, and that $T(R_2) = 21 = b$ is the number of blocks in the design. □

For ease of exposition in our examples, we take the ten designs listed in Table 8.1 as being all the distinct 2-$(7,3,3)$ designs, and find smallest defining sets under this assumption. That is, we use $\mathcal{D}$ instead of $\mathcal{D}^*$. So we will find, for each of the ten designs, the smallest number of blocks needed to distinguish it from the other nine.

Remark: This is a type of establishing set, where the information we are given includes a list of all designs which are 'allowable' completions.

8.3 Discriminating sets

Suppose that we have a distinguished design $D \in \mathcal{D}^*$, and we wish to find a defining set for D. That is, we wish to uniquely identify D among all the designs in $\mathcal{D}^*$, using only a list of blocks drawn from D. Let D' be any design in $\mathcal{D}^* \setminus \{D\}$. How can we discriminate in favour of D, and against D'? Obviously, we must do this on the basis of the differences between D and D'.

DEFINITION 8.3: *Let* $D = \{(s_0, m_0), \ldots, (s_i, m_i)\}$ *and* $D' = \{(t_0, n_0), \ldots, (t_j, n_j)\}$ *be distinct designs, and suppose that* $C = D \sqcap D'$ *is the set of melts common to both* D *and* D'*. Then* $DS(D, D') = (D \setminus C, D' \setminus C)$*, is the* ***discriminating set*** (DS) *for the pair* D *and* D'*.*

EXAMPLE 8.4: The top half of Figure 8.1 shows the discriminating sets for some pairs of designs from $\mathcal{D}$. The blocks are in lexicographic order, reading down the columns. Blocks in both designs, but with different multiplicities, are on the same row. Thus, for example, $DS(R_8, R_9) = (\{(024, 1)$, (056,2), (126,1), (145,2)}, {(026,1), (045,1), (056,1), (124,1), (145,1), (156,1)}). □

REMARK: Note that DS are mutually t-balanced. If the designs are simple, then a DS is a trade. If the designs are non-simple, the two parts of a DS need not be disjoint; so it need not be a trade, although it does contain a trade.

Given $DS(D, D')$, the knowledge that a particular melt of $DS(D, D')$ is in, or is not in, the design we are attempting to identify enables us to discriminate between D and D'. Note that we need both the block and its multiplicity to discriminate. We may speak loosely, and say that $D \setminus C$ is a discriminating set of D, and that it discriminates **for** D and **against** D'. Any melt of $D \setminus C$ is said to discriminate for D and against D'.

EXAMPLE 8.5: {(024,1), (056,2), (126,1), (145,2)} is a discriminating set of R_8, and it discriminates for R_8 and against R_9. The melt $(145, 2)$ discriminates for R_8 and, in fact, identifies it uniquely among the designs of $\mathcal{D}$. The melt $(146, 2)$ discriminates for R_2, R_3, R_4, R_6 and R_7 and against the other five designs. □

If our aim is to find defining sets of designs, DS are not quite what we require. Defining sets consist solely of lists of blocks of the design, with no multiplicity information. If a block occurs m times in a defining set, it must occur at least m times in the completed design, but may occur more than m times. Further, the multiplicities of blocks in a DS are sometimes more than is required to discriminate between the particular pair of designs.

EXAMPLE 8.6: The set {012, 012, 012, 056, 056, 156, 236, 236, 236, 245, 245} is a defining set of R_2, among the designs in $\mathcal{D}^*$, but only contains the block 245

Figure 8.1: Some example DS and RDS

I	R_1	R_1	R_2	R_2	R_8	R_8	R_9
(012,1)	(012,3)	(034,3)	(034,2)	(012,3)	(012,2)	(024,1)	-
(014,1)	-	-	(035,1)	-	(013,1)	-	(026,1)
(016,1)	-	-	(046,1)	-	(024,1)	-	(045,1)
(024,1)	-	(056,3)	(056,2)	(034,2)	(034,1)	(056,2)	(056,1)
(025,1)	-	-	(134,1)	-	(126,1)	-	(124,1)
(034,1)	(034,3)	(135,3)	(135,2)	(134,1)	-	(126,1)	-
(035,1)	-	(146,3)	(146,2)	(135,2)	(135,1)	(145,2)	(145,1)
(036,1)	-	-	(156,1)	-	(136,1)	-	(156,1)
(056,1)	(056,3)			-	(145,2)		
(123,1)	-			(146,2)	(146,1)		
(125,1)	-			(156,1)	-		
(135,1)	(135,3)			-	(234,1)		
(136,1)	-			-	(235,1)		
(145,1)	-			(236,3)	(236,1)		
(146,1)	(146,3)			(245,3)	(245,1)		
(234,1)	-			-	(256,1)		
(236,1)	(236,3)			-	(346,1)		
-	(245,3)						
(246,1)	-						
(256,1)	-						
(345,1)	-						
(456,1)	-						
-	(012,2)	(034,3)	-	(012,3)	-	(024,1)	-
(014,1)	-	-	(035,1)	-	(013,1)	-	(026,1)
(016,1)	-	-	(046,1)	-	(024,1)	-	(045,1)
(024,1)	-	(056,3)	-	(034,2)	-	(056,2)	-
(025,1)	-	-	(134,1)	-	(126,1)	-	(124,1)
-	(034,2)	(135,3)	-	(134,1)	-	(126,1)	-
(035,1)	-	(146,3)	-	(135,2)	-	(145,2)	-
(036,1)	-	-	(156,1)	-	(136,1)	-	(156,1)
-	(056,2)			-	(145,1)		
(123,1)	-			(146,2)	-		
(125,1)	-			(156,1)	-		
-	(135,2)			-	(234,1)		
(136,1)	-			-	(235,1)		
(145,1)	-			(236,2)	-		
-	(146,2)			(245,2)	-		
(234,1)	-			-	(256,1)		
-	(236,2)			-	(346,1)		
-	(245,1)						
(246,1)	-						
(256,1)	-						
(345,1)	-						
(456,1)	-						

twice. The third occurrence of the block in the completed design is forced. The melt $(245, 3)$ discriminates for R_2 and against R_8. However, the knowledge that the design (that is, R_2) has at least two copies of the block 245 in it is sufficienct to discriminate in its favour and against R_8. □

Since we wish to find defining sets and, in particular, smallest defining sets, we accordingly reduce the amount of information in each DS to the minimum necessary to discriminate between a pair of designs.

DEFINITION 8.7: *Let $DS(D, D') = (M, M')$ be the discriminating set of the designs D and D'. We form the* ***reduced discriminating set*** *(RDS) of the designs D and D', $RDS(D, D') = (R, R')$, as follows. If $(s, m) \in M$ and s is not represented in M', then $(s, 1) \in R$. If s is not represented in M and $(s, m') \in M'$, then $(s, 1) \in R'$. If $(s, m) \in M$, $(s, m') \in M$, and $m > m'$ ($m \neq m'$, by definition), then $(s, m'+1) \in R$; if $m < m'$, then $(s, m+1) \in R'$.*

The rationale behind this definition should be obvious. For example, in the case $m > m'$, the knowledge that the design contains at least m' copies of the block s is not sufficient to discriminate between D and D'. On the other hand, the knowledge that it contains more than m' copies of s is sufficient to discriminate for D and against D'. We say that the first set of melts in the ordered pair $RDS(D, D')$ (that is, those melts or submelts drawn from D) is the RDS of D with respect to D'.

EXAMPLE 8.8: The bottom part of Table 8.1 gives the RDS derived from the corresponding DS in the top part of the table. Note that a melt containing a particular block can be in one or other of the pair of designs as part of an RDS, but not both. For example, although both R_2 and R_8 contain copies of the block 245, this block can only be used, as part of a defining set, to discriminate for R_2 and against R_8. In the absence of precise multiplicity information, it cannot be used to discriminate in favour of R_8 and against R_2. □

REMARK: Suppose that $m > m'$ in the melts (s, m) and (s, m') of $DS(D, D')$. Then any value in the range $m'+1, \ldots, m$ for the number of occurrences of s in a defining set is sufficient to discriminate for D and against D'.

To use RDS to find all defining sets of a design, we need the following two results, analogous to Lemmas 1.8 and 1.9 for trades.

LEMMA 8.9: *Any defining set S of a design D contains a melt, or a supermelt, of every reduced discriminating set of D.*

PROOF: We will prove the contrapositive; so, suppose $D' \in \mathcal{D}^* \setminus \{D\}$ is such that $S \subseteq D$ does not contain a melt, or a supermelt, of the RDS of D with respect to D'.

Suppose $C = D \sqcap D'$ and let $D = C \cup E$ and $D' = C \cup E'$. Note that $C \cap E = \emptyset$, $C \cap E' = \emptyset$ and $E \sqcap E' = \emptyset$. Then $DS(D, D') = (E, E')$ and $RDS(D, D') = (F, F')$, with $F \subseteq E$ and $F' \subseteq E'$. Let $E = F \cup H$ and $E' = F' \cup H'$. Now a melt $u = (s, m)$ can be in H for two reasons. If $u \in E$, and a supermelt of u is in E', then u was deleted from the DS when it was reduced. If $(s, m + m' + 1) \in E$ and $(s, m') \in E'$, then m copies of s (that is, u) were removed from the DS when it was reduced. Call these cases (α) and (β), respectively.

Now, if $S \subseteq C$, then $S \subseteq D$ and $S \subseteq D'$. Thus S is not a defining set of D, and the result follows. So let $u = (s, m) \in E$ and suppose that S contains the melt (s, m^*), for some $0 < m^* \leq m$. Now, in case (α), $u \in H$ and, by the definition of RDS, F' contains $(s, m + 1)$. In case (β), F will contain the melt (s, m_*), for some $m_* > 0$. But $m^* < m_*$, since S does not contain a melt or a supermelt of F, and $(s, m_* - 1) \in H'$, by the definition of RDS.

So, in either case, E' contains (s, m^*), or a supermelt of (s, m^*). Thus, $S \subseteq D'$, and S is not a defining set of D. ☐

Lemma 8.10: *Any submultiset S of blocks of a design D which contains a melt, or a supermelt, of every reduced discriminating set of D is a defining set of D.*

Proof: Let $D' \in \mathcal{D}^* \backslash \{D\}$ be any design distinct from D and consider $RDS(D, D')$. By assumption, S contains $u = (s, m)$, a melt of $RDS(D, D')$ that belongs to D, or contains $u = (s, m + e)$, $e > 0$, a supermelt of a melt (s, m) of $RDS(D, D')$ that belongs to D. In either case, by the definition of a reduced discriminating set, s can be in D' with multiplicity at most $m - 1$. Thus, u discriminates for D and against D' and $S \not\subseteq D'$. ☐

Together, these lemmas give the connection we require between reduced discriminating sets and defining sets.

Theorem 8.11: *A submultiset S of a design D is a defining set of D if and only if it contains a melt, or a supermelt, of every reduced discriminating set of D.* ☐

Example 8.12: Since I is the only design of $\mathcal{D}$ that contains the block 014, (014,1) is in every RDS of I and {014} is a defining set. It is obviously a smallest defining set. As noted previously, R_8 is the only design that contains at least two copies of the block 145. Thus, every RDS of R_8 contains either $(145, 1)$ or $(145, 2)$, and {145,145} is a defining set of R_8. This is a minimal defining set, but it is not a smallest one – consider {126}. ☐

8.4 An integer linear programme

In light of the results of the last section, we see that, given a list of all the RDS in all the designs in $\mathcal{D}$, we can find all the defining sets of all the designs. We wish to find smallest defining sets, and one way to do this is to construct an integer linear programme (ILP), the optimal solutions of which correspond to smallest defining sets.

The problem here is to handle those melts that have, in the RDS, multiplicities of more than one. There may be several melts in different RDS with the same block s of the design, but different multiplicities. We must have s occur the appropriate number of times in the ILP's solution S if we choose it as the representative of a particular RDS, but we must minimise the total cardinality $T(S)$ of the solution.

When working with trades, we could frame an ILP where both the coefficients and the variables were binary, and where the bounds on our inequalities were 1's. That is, a BILP. For RDS, we frame a pseudo-BILP. Here the variables and bounds are still binary, but the coefficients need not be; in our case, they can be -1, 0 or $+1$.

Recall that the blocks of the designs are indexed in lexicographic order. Now, given a melt (s, m) of an RDS R of D, suppose that the blocks equal to s in D are $b_i, \ldots, b_{i+m^*}$, for some $m^* \geq m - 1$. We use the block b_{i+m-1} as the representative of (s, m) in framing the inequality for R. Recall that we use the b_i's as variables in our ILP to stand for the presence or absence of the b_i's in our solution. Now, if (s, m) is used as part of our optimal solution, then we require $b_i = \ldots = b_{i+m-1} = 1$. To enforce this, when $m > 1$, we add the $m - 1$ inequalities $b_i - b_{i+1} \geq 0, \ldots, b_{i+m-2} - b_{i+m-1} \geq 0$ to our system. That is, we require $b_j \geq b_{j+1}$, for $i \leq j \leq i+m-2$. Thus, fixing the value of b_{i+m-1} at 1 as part of our solution forces all of the lesser indexed blocks, for the same s, to be 1 also. A value of 0 for b_{i+m-1} does not force the values for the lesser indexed blocks.

Note that the values of $b_{i+m}, \ldots, b_{i+m^*}$ are undetermined by the particular RDS R. These are 0 by default, since our objective is to minimise $T(S) = \sum b_i$, while some of them may be set to 1 by other RDS. If s is represented in several RDS, it will figure in several of the inequalities of the system, perhaps under different block numbers. Among all of these, the one with the highest indexed b_i which is set to 1 will determine the values for all the others. Those with lesser indices will be set to 1 and those with greater to 0.

Example 8.13: For the designs of $\mathcal{D}$, our objective is to minimise $\sum_{i=0}^{20} b_i$. If we are trying to find a defining set for R_8, then $RDS(R_8, R_9)$ yields the inequalities $b_3 + b_8 + b_9 + b_{13} \geq 1$, $b_7 - b_8 \geq 0$ and $b_{12} - b_{13} \geq 0$. Note that, in practice, we would list all the possible $b_i - b_{i+1} \geq 0$ inequalites once and then have a single additional inequality for each RDS. Thus, for R_8, the full set of multiplicity control inequalities is $b_0 - b_1 \geq 0$, $b_7 - b_8 \geq 0$ and $b_{12} - b_{13} \geq 0$. □

Given a solution S, not necessarily optimal, of an ILP, if no proper subset of S is also a solution, then S is called a **critical** solution. The following result, in light of the theorem of the last section, and the formulation of the pseudo-BILP, is obvious.

Theorem 8.14: (1) *A submultiset S of blocks of a design D is a defining set of D if and only if it is a solution to the pseudo-BILP obtained from the complete list of reduced discriminating sets of D in the manner prescribed;*
(2) *S is a minimal defining set of D if and only if S is a critical solution to the pseudo-BILP;*
(3) *S is a smallest defining set of D if and only if it is an optimal solution to the pseudo-BILP.* □

When using this theorem, we must take care when moving between designs and defining sets on the one hand, and the pseudo-BILP on the other. In the former, we do not distinguish between identical blocks in the design. In the latter, each block is assigned a unique number, and is distinguished from all other blocks. Thus, given a smallest defining set S of D, when we come to select the b_i to represent the blocks of S, we must use the lowest indexed blocks, to match the fashion in which we constructed the pseudo-BILP. If we used as representatives the highest indexed blocks, it would be possible for S to be a smallest defining set without, formally, being an optimal solution to the pseudo-BILP as presented.

Given the design D, it is possible to form all possible multiplicity control inequalities of the form $b_i - b_{i+1} \geq 0$ without knowledge of any of the RDS in D. It may be the case that not all of these inequalites are forced on us by the RDS in the design. Inclusion or not of any unforced inequalities does not affect the above result, provided we recall that the separate identity of blocks is lost when moving from the pseudo-BILP to defining sets.

Example 8.15: There are no examples of unforced inequalities among the RDS of the designs of $\mathcal{D}$. The full pseudo-BILP for R_2 is given in Figure 8.2. One optimal

FIGURE 8.2: The pseudo-BILP for R_2 (in $\mathcal{D}$)

Minimise

$$\sum_{i=0}^{20} b_i,$$

subject to

$$\begin{array}{rcl}
b_0 - b_1 & \geq & 0, \\
b_1 - b_2 & \geq & 0, \\
b_3 - b_4 & \geq & 0, \\
b_7 - b_8 & \geq & 0, \\
b_{10} - b_{11} & \geq & 0, \\
b_{12} - b_{13} & \geq & 0, \\
b_{15} - b_{16} & \geq & 0, \\
b_{16} - b_{17} & \geq & 0, \\
b_{18} - b_{19} & \geq & 0, \\
b_{19} - b_{20} & \geq & 0,
\end{array}$$

and

$$\begin{array}{rcl}
b_1 + b_4 + b_6 + b_8 + b_9 + b_{11} + b_{13} + b_{14} + b_{16} + b_{18} & \geq & 1, \\
b_5 + b_6 + b_9 + b_{14} & \geq & 1, \\
b_2 + b_8 + b_{11} + b_{17} & \geq & 1, \\
b_9 + b_{14} + b_{17} + b_{20} & \geq & 1, \\
b_4 + b_8 + b_{11} + b_{13} + b_{16} + b_{19} & \geq & 1, \\
b_2 + b_5 + b_6 + b_9 + b_{17} + b_{20} & \geq & 1, \\
b_2 + b_5 + b_6 + b_9 + b_{11} + b_{14} + b_{16} + b_{19} & \geq & 1, \\
b_2 + b_4 + b_9 + b_{11} + b_{13} + b_{14} + b_{16} + b_{19} & \geq & 1, \\
b_2 + b_4 + b_8 + b_9 + b_{11} + b_{13} + b_{16} + b_{19} & \geq & 1.
\end{array}$$

solution to this system is $b_7 = b_8 = b_9 = 1$, so $\{056,\ 056,\ 134\}$ is a smallest defining set for R_2, with respect to the other nine designs in $\mathcal{D}$. □

REMARK: Note that there are exactly $b - b^*$ possible multiplicity inequalities. Thus, in practice, the impact of these on the number of inequalities in the ILP is negligible. However, they are responsible for transforming what would be a BILP into a pseudo-BILP.

EXAMPLE 8.16: The starred blocks of the designs in Table 8.1 correspond to smallest defining sets obtained via optimal solutions to the pseudo-BILP. □

8.5 Minimal discriminating sets

Recall that $\mathcal{D}^*$ is the set of all distinct designs with given parameters, and that $|\mathcal{D}^*| = N$. Now $D \in \mathcal{D}^*$ has an RDS with each of the other $N - 1$ designs. However, as in the case of trades, these RDS need not be distinct. Thus, $N - 1$

is an upper bound on the number of distinct RDS of D, where distinct is taken to mean "containing a different set of melts or submelts of D".

EXAMPLE 8.17: Each of the ten designs of $\mathcal{D}$ can have at most nine distinct RDS. For an example where this upper bound is not achieved, note that the RDS of R_6 with respect to R_1 equals that with respect to R_4. That is, {(013,1), (026,1), (045,1), (124,1), (156,1), (235,1), (346,1)}. Of course, the RDS of R_1 and R_4 with respect to R_6 are different. □

As in the case of trades, the number of distinct RDS in a design could be very large. This could cause the running time to solve the pseudo-BILP to become too long to be practical. Are all the RDS in a design necessary to find all the defining sets of a design, or can we minimise the collection in a similar fashion to trades?

Suppose that the collection of RDS of D contains the distinct RDS R' and R''. Suppose further that, for every melt (s, m') in R', the melt (s, m'') is contained in R'', and that $m' \geq m'' > 0$. Then any subset S of D that contains a melt, or a supermelt, of R' will also contain a melt, or a supermelt, of R''. Of course, the converse will not be true in general.

We say that the RDS R' **covers** the RDS R''. Given a collection of RDS, any RDS in the collection that is not covered by another RDS is said to be **minimal**. A collection of RDS where no RDS is covered by another RDS is said to be **minimised**.

EXAMPLE 8.18: The RDS of R_8 with respect to R_9, {(024,1), (056,2), (126,1), (145,2)}, covers the RDS of R_8 with respect to R_3, {(013,1), (024,1), (056,2), (126,1), (136,1), (145,1), (234,1), (235,1), (256,1), (346,1)}. □

REMARK: If all the designs are simple, then RDS are trades, and the definition of minimal accords with our previous usage.

The next result follows immediately from the definition of the covering relation.

LEMMA 8.19: *A submultiset S of a design D is a solution to the pseudo-BILP obtained from the collection of all reduced discriminating sets of D if and only if it is a solution to the pseudo-BILP obtained from the minimised collection of all reduced discriminating sets of D.* □

Suppose that R and R' are distinct RDS, and that R covers R'. Obviously, $|R| \leq |R'|$, but both $|R| < |R'|$ and $|R| = |R'|$ are possible. We can say nothing about the relative values of $T(R)$ and $T(R')$. However, we can minimise a list of RDS using a simple variation of the technique of Subsection 6.2.1. We order RDS by cardinality – that is, number of melts – and process them in increasing order. We first compare each RDS of a given size with the list of smaller (minimal) RDS, eliminating any that are covered. We now compare the RDS of a given size that remain among themselves, eliminating any that are covered. The remaining RDS are the minimal RDS of this cardinality.

REMARK: The covering relation is a partial order on RDS, and so a collection of RDS forms a *partially ordered set* or *poset* [3]. The problem of finding minimal RDS is thus equivalent to the problem of finding minimal elements in a poset; similarly for trades. Trades can be *ranked* by their cardinality, with two trades of the same rank being incomparable. Neither $|\cdot|$ nor $T(.)$ can be used to rank RDS, hence the more complicated algorithm for minimisation.

8.6 Conclusion

The algorithm given in Chapter 6 can be modified, in the obvious manner, to enumerate RDS instead of trades; the complexity remains $O(v!n^2b^2)$. With a complete list of the RDS in a design to hand, the appropriate collection of RDS can be extracted, minimised if necessary, and a pseudo-BILP generated. The utility `opbdp` that we use is actually intended for solving pseudo-BILP, albeit of a more general form than ours. It can thus be used to solve the pseudo-BILP, generated from lists of RDS, of the form described in Section 8.4. We present some results obtained by this technique in the next chapter.

Chapter 9

Results on non-simple designs

In this chapter, lists of all the minimal RDS in a design are used to obtain information regarding its defining sets. We consider three series of t-(v, k, λ) designs, where we fix t, v and k, and vary λ. We are interested in investigating how good the bound of Lemma 2.31 is for decomposable designs, and the size of smallest member and class defining sets. A simple algorithm to generate a series of designs is described. We prove the value of $|d_s D|$ for all 2-$(6, 3, \lambda)$ designs, the first time this has been done for an infinite family of designs. A method of generating the 3-$(8, 4, \lambda)$ designs from the 2-$(7, 3, \lambda)$ is developed. A good upper bound on $|d_s D|$ for these two families of designs is proved. We also revisit the 2-$(9, 3, 2)$ designs, and obtain some new information on their various defining sets.

9.1 The 2-$(6, 3, \lambda)$ designs

For 2-$(6, 3, \lambda)$ designs, $\lambda_* = 2$ and $\lambda^* = 4$. The 2-$(6, 3, 2)$ design is unique and is the only indecomposable design [56]. A fifth degree polynomial in λ is known which gives the number of different designs for each even λ, see [27]. A smallest defining set of the 2-$(6, 3, 2)$ design has three blocks, see [46]. A smallest defining set is isomorphic to $\{012, 013, 024\}$, is unique, and occurs 30 times in the design. It is the only minimal defining set of the design.

For our 2-$(6, 3, 2)$ design, we will take the smallest design in lexicographic order. We call this design P_0, and list its blocks in Figure 9.1. Now $|\text{aut}(P_0)| = 60$, and so there are twelve distinct designs. The eleven designs distinct from P_0 are labelled $P_1, \ldots, P_9, P_a, P_b$. A permutation generating each of these from P_0 is given in Figure 9.1.

With this list of designs to hand, it is a simple matter to generate all different designs for a particular λ as unions of P_i's, using `nauty` to eliminate isomorphs. We can now use lists of RDS from these designs to calculate the sizes of various smallest defining sets. The results, for $\lambda \leq 12$, are collected in Tables F.1–F.5 of Appendix F.

FIGURE 9.1: The 12 distinct 2-$(6,3,2)$ designs

Design	Blocks
P_0	$\{012, 013, 024, 035, 045, 125, 134, 145, 234, 235\}$
P_1	$(012345)P_0$
P_2	$(02345)P_0$
P_3	$(021345)P_0$
P_4	$(0345)(12)P_0$
P_5	$(01345)P_0$
P_6	$(0345)P_0$
P_7	$(023145)P_0$
P_8	$(045)(123)P_0$
P_9	$(045)(13)P_0$
P_a	$(0145)(23)P_0$
P_b	$(045)(23)P_0$

The first column of these tables lists the designs from Figure 9.1 used to generate the design; this decomposition is not necessarily unique. In the next column the support size of the design is given. In [64] the possible support sizes of 2-$(6,3,\lambda)$ designs, and the smallest b for which a design with each b^* exists, were given; our results match these. The next column gives the order of $\text{aut}(D)$ for each design. This data matches that in [89], where the number of different designs for each automorphism group order is calculated. Note that none of the listed designs is rigid; the first rigid designs have $\lambda = 14$.

The final three columns give the sizes of smallest member defining sets, class defining sets, and defining sets. We see that the bound of Lemma 2.31 is, in fact, the actual value of $|d_s D|$.

THEOREM 9.1: *Let* $D = (V, \mathcal{B})$ *be a* 2-$(6,3,\lambda)$ *design. Then* $|d_s D| = 3\lambda/2$.

PROOF: Given the lower bound of Lemma 2.31, we need only exhibit a set of $3\lambda/2$ blocks that completes uniquely. Suppose that the points of V are $0, \ldots, 5$, and note that $b = 5\lambda$ and $r = 5\lambda/2$. Let $\mathcal{B} = X \cup Y \cup Z$, where X is the set of $3\lambda/2 = 5\lambda/2 - \lambda$ blocks that contain the point 0 but not the pair 05, Y is the set of blocks that contain the pair 05, and Z is the set of blocks that do not contain 0.

We show that X completes uniquely. Consider pairs of the form $0a$, $1 \leq a \leq 4$, that occur in X. If such a pair does not occur λ times, then a must occur in a block of the form $05a$ the appropriate number of times. Thus the blocks of Y are forced. Now the foundation size $f(Z)$ is 5, and a $(3,2)$ trade must have foundation at least $k+t+1 = 6$. Thus Z cannot contain any $(3,2)$ trades, and $X \cup Y$ completes uniquely. So X is a defining set. □

For class defining sets, we note that $|c_sD| = |d_sD|$ if 4 divides λ. If not, then $|c_sD|$ is either $|d_sD|$ or $|d_sD| - 1$. For the member defining sets, the results vary considerably, with many intriguing patterns. We prove two simple results.

LEMMA 9.2: *If D is a multiple of the 2-$(6,3,2)$ design E, then $|m_sD| = |d_sE| = 3$.*

PROOF: Since E is simple, and has a smallest defining set of size three, any block or pair of blocks from D obviously has more than one completion to designs isomorphic to D. A 3-block smallest defining set of E completes uniquely to E and then, since D is known to be a multiple of a 2-$(6,3,2)$ design, D is forced. □

LEMMA 9.3: *If D is a multiple of the full 2-$(6,3,4)$ design, then $|m_sD| = 0$.*

PROOF: The full design, and its multiples, has $|\text{aut}(D)| = v!$, and so there is only one distinct design in the isomorphism class. □

9.2 The 2-$(7,3,\lambda)$ designs

For 2-$(7,3,\lambda)$ designs, $\lambda_* = 1$ and $\lambda^* = 5$. The smallest defining set sizes for the designs with $\lambda \leq 3$ were found in [47]; we extend these results to $\lambda = 4$, and also find the sizes of smallest member and class defining sets. There are only two indecomposable 2-$(7,3,\lambda)$ designs for $\lambda \leq 4$, one each for $\lambda = 1$ and $\lambda = 3$ [56]. In fact, these are the only indecomposable 2-$(7,3,\lambda)$ designs, see [81].

For the indecomposable 2-$(7,3,1)$ design – the Fano plane – we take the cyclic design developed from the starter block 013. For the indecomposable 2-$(7,3,3)$ design, we remove the two disjoint Fano planes developed cyclically from 013 and 023 from the full design. We label these designs 1.0 and 3.0 respectively, and list the blocks of 1.0 in Figure 9.2.

Now $|\text{aut}(1.0)| = 168$, and so there are 30 distinct designs. The 29 designs distinct from 1.0 are labelled $1.1, \ldots, 1.29$. A permutation generating each of these from 1.0 is given in Figure 9.2. For 3.0, $|\text{aut}(3.0)| = 42$, and so there are 120 distinct designs; we do not list these, since our designs are constructed using at most one copy of the 2-$(7,3,3)$ design, and this copy is always 3.0.

The non-isomorphic designs obtained by taking unions of the indecomposable designs are tabulated in Tables F.6–F.8 of Appendix F. Note that the 2-$(7,3,4)$ design marked with a (+) also has a decomposition into four 2-$(7,3,1)$ designs. The second column of these tables give the support of the design and the block multiplicities,

Figure 9.2: The 30 distinct 2-$(7,3,1)$ designs

Design	Blocks	Design	Blocks
1.0	$\{013, 124, 235, 346, 450, 561, 602\}$	1.15	(0456)(132)1.0
1.1	(023456)1.0	1.16	(01456)(23)1.0
1.2	(0213456)1.0	1.17	(0456)(23)1.0
1.3	(03456)(12)1.0	1.18	(012456)1.0
1.4	(013456)1.0	1.19	(02456)1.0
1.5	(03456)1.0	1.20	(021456)1.0
1.6	(0312456)1.0	1.21	(0456)(12)1.0
1.7	(02456)(13)1.0	1.22	(01456)1.0
1.8	(0231456)1.0	1.23	(0456)1.0
1.9	(0456)(123)1.0	1.24	(0234156)1.0
1.10	(031456)1.0	1.25	(034156)1.0
1.11	(0456)(13)1.0	1.26	(0324156)1.0
1.12	(0132456)1.0	1.27	(04156)(23)1.0
1.13	(032456)1.0	1.28	(024156)1.0
1.14	(0321456)1.0	1.29	(04156)1.0

with the value of n_i giving the number of k-subsets that occur i times in the design. The third column gives the automorphism group order for the designs. Note that the data in the second and third columns distinguishes among all the designs, except for the 2-$(7,3,4)$ pair with block multiplicity vector (16,6,0,0) and $|\text{aut}(D)| = 2$.

The values of the various smallest defining sets are given in the final three columns. Note that if a design has a decomposition into Fano planes, then $|d_sD| = 3\lambda$. However, if all decompositions of a design necessarily contain a copy of the indecomposable 2-$(7,3,3)$ design, then the lower bound of Lemma 2.31 need not be tight. Smallest class defining set sizes usually equal $|d_sD|$, although $|d_sD| - |c_sD| = 2$ for one of the 2-$(7,3,4)$ designs. As for the 2-$(6,3,\lambda)$ designs, the results for $|m_sD|$ exhibit considerable variability, and results similar to Lemmas 9.2 and 9.3 are easily proved.

Lemmas 2.25 and 2.31 between them imply that, for any 2-$(7,3,\lambda)$ design, $1/3 \leq \mu \leq 4/7$. We prove a better upper bound, which shows that any 2-$(7,3,\lambda)$ design can be defined by less than half its blocks.

Theorem 9.4: *If* D *is a* 2-$(7,3,\lambda)$ *design, then* $|d_sD| \leq 16\lambda/5$.

Proof: Let $V = \{0,\ldots,6\}$, and suppose that $D = (V, \mathcal{B})$ is a 2-$(7,3,\lambda)$ design, with $b = 7\lambda$ and $r = 3\lambda$. Let $\mathcal{B} = X \cup Y$, where X is the set of blocks that do not contain the point 0. As in the proof of Lemma 2.25, X is a defining set of D, in $b - r = 4\lambda$ blocks. We show that X contains a set of at least $4\lambda/5$ blocks that can be deleted without destroying its unique completion property.

Consider the 3λ blocks in Y. These contain a total of 3λ pairs from $V \setminus \{0\}$. Since there are $\binom{6}{2} = 15$ possible pairs, there must be a pair, say ab, that occurs at most $3\lambda/15 = \lambda/5$ times in Y. Since each pair occurs λ times in D, there must be at least $4\lambda/5$ blocks in X that contain ab. Let the set of blocks in X that contain ab be Z, and consider $X \setminus Z$.

Now X is a 1-$(6, 3, 2\lambda)$ design, and the blocks of Z have a common pair, so, by Theorem 5.11, $X \setminus Z$ completes uniquely to X. Hence, $X \setminus Z$ is a defining set of D, and $|d_s D| \leq 4\lambda - 4\lambda/5 = 16\lambda/5$. □

COROLLARY 9.5: *For any* 2-$(7, 3, \lambda)$ *design,* $1/3 \leq \mu \leq 16/35$. □

Theorem 9.4, together with Lemma 2.31, is sufficient to prove the value of $|d_s D|$ for all 2-$(7, 3, \lambda)$ designs which are decomposable into Fano planes and have $\lambda \leq 4$.

9.3 The 3-$(8, 4, \lambda)$ designs

The 3-$(8, 4, \lambda)$ designs are self-complementary, and can all be obtained by extending uniquely the 2-$(7, 3, \lambda)$ designs by complementation [78]. It is easy to see that the extension of an indecomposable 2-$(7, 3, \lambda)$ design is an indecomposable 3-$(8, 4, \lambda)$ design, while any restriction of an indecomposable 3-$(8, 4, \lambda)$ design is an indecomposable 2-$(7, 3, \lambda)$ design. So there are precisely two indecomposable 3-$(8, 4, \lambda)$ designs, one for $\lambda = 1$ and one for $\lambda = 3$.

We could proceed as for the 2-$(7, 3, \lambda)$ designs, and enumerate the 3-designs by taking unions of the two indecomposable designs. However, since each 2-design extends uniquely, the number of 3-$(8, 4, \lambda)$ designs is bounded above by the number of 2-$(7, 3, \lambda)$ designs. Hence, the 3-designs can be enumerated simply by extending the 2-$(7, 3, \lambda)$ designs and then using `nauty` to eliminate isomorphs. For $1 \leq \lambda \leq 3$, all the extensions are non-isomorphic, while for $\lambda = 4$, the 35 extensions yield 31 non-isomorphic 3-$(8, 4, 4)$ designs – first enumerated in [68].

The smallest defining set sizes for the designs with $\lambda \leq 3$ were found in [47]; we extend these results to $\lambda = 4$. Of course, given Theorem 2.35, we know that a 3-$(8, 4, \lambda)$ design and its restrictions have the same values of $|d_s D|$. However, we are also interested in the values of smallest member and class defining sets. The results are presented in Tables F.9–F.11 of Appendix F, with each design listed being the extension of the corresponding design in Tables F.6–F.8. Note the doubling of b^*

and the entries in the block multiplicity vector, caused by the extension process (cf. [78]). For the 3-$(8,4,4)$ designs we indicate, in the first column of Table F.9, which extensions of the 2-designs are isomorphic.

Recall that all 8 of the restrictions of a homogeneous 3-$(8,4,\lambda)$ design are isomorphic. All 3-$(8,4,2)$, 3-$(8,4,3)$, and 27 of the 31 3-$(8,4,4)$ are homogeneous. For each of these designs all three smallest defining sets have the same values in the 2-design and its extension, and $|\mathrm{aut}(D)|$ for the 3-design is eight times that of the 2-design. For the inhomogeneous 3-designs, the values of $|d_sD|$ for the 3-design and its various restrictions are equal, as required by Theorem 2.35. However, the values of $|m_sD|$ in the restrictions need not match, nor need the relationship between the values of $|\mathrm{aut}(D)|$ hold.

Again, results similar to Lemmas 9.2 and 9.3 are easily proved. Further, since the 2-designs extend uniquely, the defining set of the 2-designs described in Theorem 9.4 is also a defining set of the extensions to 3-designs. Since the extension process doubles the number of blocks, we have $1/6 \leq \mu \leq 8/35$ in any 3-$(8,4,\lambda)$ design.

9.4 The 2-$(9,3,2)$ designs

Recall that the $STS(9)$ is unique, and has $|d_SD| = 4$, see [46]. There are precisely 36 different 2-$(9,3,2)$ designs, see [96, 87]. Smallest defining sets are given in [71]. We take the designs, and their properties, from these listings, labelling them K_1 to K_{36}. Lists of RDS in the designs were used to verify the sizes of smallest defining sets, and were also used to find the sizes of smallest member and class defining sets. Additionally, the sizes of smallest establishing sets were found, where the establishing information is a set of blocks together with the knowledge of whether or not the design is simple.

The results are collated in Table 9.1. The second column gives the support size of the design; if this is 24 the design is simple. The next column gives the single transpositions of the system; if none, the design is STF. The fourth column indicates whether or not the design is decomposable, while the fifth indicates the smallest α for which the design is α-resolvable.

Given a 3-subset $\{a,b,c\}$, we can regard it as being ordered cyclically and containing the three ordered pairs (a,b), (b,c) and (c,a). A collection of such cyclically ordered blocks that contains each ordered pair from V exactly once is called a **Mendelsohn**

Table 9.1: Results for the 2-$(9, 3, 2)$ designs

Design	b^*	STF	Dec	α	Men	$\lvert A\rvert$	$\lvert m_sK_i\rvert$	$\lvert c_sK_i\rvert$	$\lvert d_sK_i\rvert$	$\lvert$Est$\rvert$
K_1	12	Yes	Yes	1	1	432	4	8	8	8
K_2	18	(12)	Yes	1	1	24	5	8	8	8
K_3	20	Yes	Yes	1	1	32	6	8	8	8
K_4	20	Yes	No	2	1	8	6	9	9	9
K_5	20	Yes	No	2	0	24	6	8	8	8
K_6	21	Yes	Yes	1	0	6	6	8	8	8
K_7	22	Yes	Yes	1	0	4	6	8	8	8
K_8	22	Yes	No	2	0	2	7	9	9	8
K_9	22	Yes	No	2	1	2	6	9	9	8
K_{10}	22	Yes	No	2	1	4	6	9	9	8
K_{11}	22	Yes	No	2	1	2	6	9	9	9
K_{12}	22	Yes	No	2	0	8	6	9	9	9
K_{13}	22	Yes	No	2	0	4	6	10	10	9
K_{14}	21	Yes	Yes	1	1	108	5	9	9	9
K_{15}	23	Yes	Yes	1	0	4	7	9	9	8
K_{16}	23	Yes	No	2	0	2	7	8	8	8
K_{17}	23	Yes	No	2	0	1	7	8	9	8
K_{18}	23	Yes	No	2	0	2	7	9	9	8
K_{19}	23	Yes	No	2	1	4	6	9	9	8
K_{20}	23	Yes	No	2	0	3	6	9	9	8
K_{21}	23	Yes	No	2	0	1	6	9	9	8
K_{22}	23	Yes	No	2	2	6	6	10	10	9
K_{23}	24	Yes	Yes	1	0	18	6	9	9	8
K_{24}	24	Yes	No	2	0	6	6	9	9	7
K_{25}	24	Yes	No	2	1	8	6	8	8	8
K_{26}	24	Yes	No	2	0	1	7	9	9	8
K_{27}	24	Yes	No	2	0	1	7	9	9	8
K_{28}	24	Yes	No	2	0	1	7	9	9	8
K_{29}	24	Yes	Yes	1	1	6	7	8	8	8
K_{30}	24	Yes	No	2	1	6	7	10	10	9
K_{31}	24	Yes	No	2	1	2	6	9	9	8
K_{32}	24	Yes	No	2	0	2	6	7	8	8
K_{33}	24	Yes	No	2	0	2	6	9	9	9
K_{34}	24	Yes	No	2	0	8	6	10	10	9
K_{35}	24	(23), (67)	No	2	1	80	5	9	9	9
K_{36}	20	Yes	No	2	2	24	6	10	10	10

triple system of order v (abbreviated $MTS(v)$). Two $MTS(v)$ are **equivalent** if they are isomorphic or if reversing the cyclic order of all triples in one system gives the other. If the order is ignored, then an $MTS(v)$ is obviously a 2-$(v, 3, 2)$ design. In the sixth column of the table the number of inequivalent $MTS(9)$ that each 2-$(9, 3, 2)$ design admits is given.

The seventh column of Table 9.1 gives the order of $\mathrm{aut}(K_i)$. The last four columns give, respectively, the sizes of smallest member defining sets, class defining sets, defining sets, and simple/non-simple establishing sets. Note that three of the decomposable designs do not achieve the bound of eight from Lemma 2.31, and that $|c_sK_i| = |d_sK_i|$ in all but two cases. For the establishing sets, $|d_sK_i| - |\mathrm{Est}|$ equals 0, 1 and 2 in 17, 18 and 1 cases respectively. There is no overt relationship between the sizes of the various smallest defining sets and any of a design's properties.

9.5 Conclusions

The theory of RDS was developed specifically to address the problem of investigating defining sets in non-simple designs. In this chapter, we have demonstrated that the theory can be successfully applied to obtain results on defining sets. There are other designs to which the technique could be applied, but RDS themselves do not seem to be of any independent interest.

The techniques used to enumerate the 2-$(7, 3, \lambda)$ and 3-$(8, 4, \lambda)$ designs are capable of considerable refinement. However, they are only applicable if the indecomposable designs are known, and not too numerous.

Inspired by the computational results, new theoretical bounds on the sizes of the various defining sets for some of the designs were obtained. The techniques used can be generalised to other designs.

The lower bound on $|d_sD|$ from Lemma 2.31 is not tight in general. Further work is needed to establish the conditions under which it is tight, and to derive a better bound in those cases where it is not.

CHAPTER 10

Completing partials

In this chapter we consider the problem of completing a partial. That is, given a collection S of not necessarily distinct k-subsets of a v-set V, with $0 \leq |S| \leq b$, list all distinct t-(v, k, λ) designs that contain S.

The utility used to date to complete designs is the programme `complete`, as described in [18]. This programme is general purpose, in that it can handle completions to designs with any parameters. We describe a new completion programme, using a heuristic specific to the case $\lambda = 1$; that is, to Steiner designs. We describe the paradigm common to both programmes, and compare their performances. We also discuss the various uses to which a 'fast' completion routine can be put.

Some of the material from this chapter, and from Sections 11.1 and 11.2 of the next chapter, has been presented in the Technical Report [98] or has been published [99].

10.1 The basic paradigm

The approach used by both of the completion routines is a standard backtrack search. This is effectively a depth-first traversal of a tree of potential solutions, where we do not explore any sub-trees that we know cannot contain a solution. In this search, the k-subsets of V are ordered (usually in the lexicographic order induced by the standard numeric ordering on the points) and, at each node of the tree, we iterate over those blocks that come after that chosen at the parent of the current node, making a recursive call to the search routine for each choice of block. This ensures that all potential completions are processed and that no solution is processed more than once.

Of course, the search-space is exponential in size, with the branching factor at each node being the number of blocks available, and the depth of the tree being $b - |S|$. Note that the branching factor is not constant, and can be as bad as $\binom{v}{k}$ in non-simple designs. In developing completion routines our main efforts are directed at devising heuristics to cut down the size of the tree and the portion of it that is traversed. We do this by decreasing the tree's depth, by reducing its branching factor, or by backtracking sooner.

For the completion routines being described, the main strategy is to maintain two lists of blocks, called the **feasible** list and the **required** list. These lists are updated dynamically as the completion proceeds. The feasible list consists of those k-subsets of V from which the remainder of the completion must be drawn. That is, any block which cannot appear in the remainder of the completion has been deleted from the feasible list. The required list is a subset of the feasible list, and has the property that at least one of its members must be in the remainder of a completion.

At each stage, the required list is formed from the feasible list, and then the next block in the completion is iterated over the required list. For each choice, the feasible list is updated, and then a recursive call is made to the search routine. Good heuristics are those that keep list sizes as small as possible, and which provide information to assist in backtracking.

By keeping track of the list sizes, the number of undetermined blocks remaining in the partial, and counts of the number of times that u-subsets of V, for some $1 \leq u \leq t$, have appeared in the partial, we can implement backtracking and **forcing**. If the lists are too small, then we backtrack. As an example of forcing, suppose that the required list is produced by tracking the highest frequency u-subset that has not appeared λ_u times in the partial so far. Then it may be the case that the size of the required list is exactly right to bring the count of this particular u-subset up to λ_u. In this case either all the blocks in the required list can be added to the design – that is, they are forced – or else completion of the current partial is not possible and we backtrack. Forced blocks can be regarded as either reducing the depth of the current subtree or reducing the branching factor of the current node to one.

10.2 The `complete` routine

The `complete` routine is described in detail in [18], with some of the results obtained using it being given in [17]. `Complete` can cope with arbitrary parameter sets (provided that v is less than the machine's word size) and can handle pointwise as well as blockwise completions. We consider only blockwise completions, and outline the heuristics used.

The feasible list is called the *hitlist*. It is generated initially by picking all k-subsets of V that do not conflict with the partial supplied. If the design is flagged as being simple or if $\lambda = 1$ (which implies that the design is simple) the feasible list is updated as blocks are added to and removed from the partial, otherwise it remains unchanged throughout the completion. The required list is called the *options* list. It is generated from the feasible list using the most frequently occurring t-subset that has not occurred λ times. All blocks from the feasible list that contain the t-subset are put on the required list, provided they do not conflict with the current partial.

REMARK: Note that producing the required list in this way means that completions are not produced in lexicographic order. We also have to take care when updating the feasible list for each choice of block from the required list to ensure that each completion will be produced once only and that none are missed. The `comp03` and `comp04` routines described in the next section also exhibit this behaviour.

The main control structures are lists of the blocks in the current partial and lists of the frequencies of occurrence of the $(t-1)$ and t-subsets of V in the current partial; both lists are maintained in order of decreasing frequency. There are also some housekeeping lists to track blocks which have been tried or removed from lists.

The lists of $(t-1)$ and t-subsets are used to attempt block forcing at each stage. This is done by looking for $(t-1)$-subsets which need to occur again, and adding elements which give t-subsets which can occur again. If exactly $k-t+1$ such elements exist, a block which must be in the design has been constructed. If this block is not on the hitlist, or conflicts with the current partial, then we backtrack. Otherwise, the block is added to the design, and the whole process is repeated until all $(t-1)$-subsets have been checked. If no such blocks can be generated then no forcing is possible, and we simply iterate over the required list.

`Complete` allows the user to supply information about the design to help speed up completions. The design (that is, all designs with the given parameters) can be flagged as simple or linked. Additionally, if the partial is not a defining set, a limit on the number of completions produced can be set.

10.3 The $\lambda = 1$ routines

Suppose that we are given a parameter set with $\lambda = 1$. Then any designs must be simple. However, we can say more. No two distinct blocks in the design can intersect in t or more points, else a t-subset would be repeated. In the terminology of Section 5.3, a Steiner design is $(k-t+1)$-simple. When completing, we ensure that at all times our feasible list has the property that no block in it intersects a block in the current partial in more than $t-1$ points. We do not maintain any count whatsoever of the frequencies of occurrence of the t-subsets; the **intersection heuristic** just described ensures that each t-subset occurs at most once in any completion. Thus, completions automatically contain each t-subset exactly once.

If, at any time, the feasible list becomes too small to supply the remaining blocks in the current partial, then we backtrack. If the feasible list is exactly the correct size to fill the remaining blocks, and the feasible list itself is $(k-t+1)$-simple, then completion is forced, else we backtrack. Note that, in general, pairs of blocks in the feasible list can intersect in t or more points.

Calculating intersections, and finding their cardinality, can be done efficiently and the intersection heuristic has been found to be very effective in reducing the size of the feasible list. Since the required list is drawn from the feasible list, it too is substantially reduced in size.

Two versions of the routine were written, differing in the heuristic used to produce the required list. Programme `comp03` uses 1-subset counting (that is, element multiplicities), while programme `comp04` counts $(t-1)$-subsets. The required list contains those members of the feasible list that contain the most frequently occurring 1-subset (resp. $(t-1)$-subset) that has not occurred λ_1 (resp. λ_{t-1}) times.

REMARK: When $t = 2$ the heuristics of `comp03` and `comp04` are the same. When $t > 2$, we expect `comp04` to be the better of the two heuristics. It is easier to code 1-subset counting than $(t-1)$-subset counting; it was originally hoped that this simpler code would yield a faster programme for the case $t = 2$.

We keep track of the size of the required list, in relation to λ_1 or λ_{t-1} as appropriate, using this to implement forcing or backtracking if possible. Otherwise we iterate over the required list, adding the current iterate to the partial. The new feasible list is generated from the old by removing all members that intersect the block chosen from the required list in more than $t-1$ points, and any members of the required list

Figure 10.1: A completion example for a 3-(14, 4, 1) design

```
The input partial has 12 blocks:
  0  6  9 11
  1  4 10 12
  1  6  9 13
  2  3  6 11
  2  4  9 11
  2  4  8 12
  3  4  7 10
  4  5  7 12
  4  5  8 13
  4  8  9 10
  6  7 11 12
  6  8 11 13

Completion number #0:
  1  5  6 11    <-- first added block
  4  6 10 11
  0  4 10 13
  2  4  5 10
  0  2  4  6
  1  2  4  7
  2  3  4 13
  0  1  4  5
  3  4  5 11
  4  5  6  9
  0  3  4  8
  0  4  7  9
  0  4 11 12
  ...           <-- blocks deleted
  5  9 10 11
  7  9 10 13
  8  9 11 12
 10 11 12 13

Single completion - partial is 3-(14,4,1) defining set
```

that precede the currently chosen block from the required list. (This last is required to prevent multiple counting of completions.)

Remark: The technique of picking only a single block at a time from the required list may not be the most effective one. If m 1-subsets (resp. $(t-1)$-subsets) remain to be placed in the partial, and the required list has $n > m$ blocks in it, it might be more effective to generate and test all $\binom{n}{m}$ ways of filling out the requisite count of 1-subsets (resp. $(t-1)$-subsets) from the required list. Of course, the iteration, recursion, forcing and backtracking do this implicitly, since once a 1-subset (resp. $(t-1)$-subset) becomes the most frequently occurring one that has not occurred λ_1 (resp. λ_{t-1}) times, it remains so until it has occurred λ_1 (resp. λ_{t-1}) times. However, it might be more efficient to do this explicitly.

Example 10.1: One utility constructed using `comp04` was `cad04`. This simply completes partials to designs in all possible ways, in a similar fashion to `cad` [18], which uses `complete`. Figure 10.1 illustrates how `cad04` processed a twelve block

FIGURE 10.2: The set cardinality function

```
int card(SET ss)                       /* SET is type 'unsigned int' */
  {
  int temp = 0;
  while (ss != 0)
    { temp++;  ss = AND(ss,ss-1); }    /* AND is bit-wise '&' */
  return temp;
  }
```

defining set of one of the four 3-(14,4,1) designs. The search-tree generated had 383 516 nodes, the total running time was 60 sec., and the single completion was produced after 18 sec. Note that $\lambda_2 = 6$, and that the only pair of points that appears in the partial four or more times is $6b$ (that is, 6 and 11), which appears four times. Thus $6b$ is used to build the first required list. After the two blocks containing $6b$ are added to the partial, the pair $4a$ now occurs four times, with every other pair occuring three times or less. Thus $4a$ is used to build the next required list. The next pair chosen is 24 (being the first pair in order that occurs three times), and so on. □

10.4 Implementation issues

The `comp03` and `comp04` routines were developed to investigate defining sets of designs. Previous workers had considered the 'smaller' designs, and were having difficulty using the existing routines for the next 'larger' designs. Thus, programme speed was very important, and efficiency of implementation took precedence over elegance. Some of the factors considered are described in this section.

As in `complete`, sets are implemented as unsigned integers. This limits v to the machine's word size, typically 32 bits, but means that basic set operations are fast and constant time. Since the intersection heuristic is predicated on the ability to find set cardinalities, an efficient means of calculating the number of elements in a set is required. We use the `card` routine shown in Figure 10.2. This uses one of the bit-counting techniques discussed in [102], and takes $O(|ss|)$ time. Although [102] also gives routines which are constant time, in our case $|ss|$ is always small – certainly less than k, and usually less than t – and `card` is very efficient.

The `complete` routine makes extensive use of dynamic data structures, in the form of linked-lists. While this allows the programme to cope with an arbitrary set of parameters, it is not necessarily the best choice from an efficiency perspective. Some points to consider are:

- ▷ Allocating and deallocating memory are operating-system functions. In the absence of failure, they can be regarded as taking a fixed time; however, they are much slower than standard arithmetic operations.
- ▷ Lists can only be accessed sequentially, and so a significant amount of time can be spent following pointer chains.
- ▷ While it is straightforward to build a sorted list, if the data items are changed it is expensive to re-sort the list.
- ▷ Modern compilers have very sophisticated inbuilt optimisers. Unfortunately, the presence of pointers restricts the optimisations possible.

For these reasons, arrays were used as the primary data structures in `comp03` and `comp04`. Array operations are very fast, and optimising compilers are very effective when dealing with arrays. The only place where sorting is needed in `comp03` or `comp04` is to find the most frequently occurring element or $(t-1)$-subset in the current partial to assist in building the required list. Instead of maintaining the arrays containing these counts sorted, we simply make a single pass over the arrays as required, and pick out the largest element.

Sorting a singly-linked list using the method adopted in `complete` is an $O(n^2)$ operation. Picking the largest element of an array can be done simply in $O(n)$ time. If the number of sorts is the same as the number of times that we need to find the maximum, as is the case in our routines, there is no advantage in maintaining the array sorted.

The only potential problem with arrays in C is that their sizes have to be predeclared. The usual procedure is to pick some maximum size, and use this for all arrays, irrespective of the actual parameters. The approach adopted for `comp03` and `comp04` is slightly different.

Since we are limited to $\lambda = 1$ and $v \leq 32$, the number of admissible parameter sets is not large. In practice, the number of blocks in a design strongly affects the running time of completion routines. If we take, say, $b < 1000$, there are only 39 admissible parameter sets. Table 10.1 lists the parameters of these 39 designs, as well as the number of different designs. This table was compiled from [14, 25]. Note that the 2-$(16, 6, 1)$ and 2-$(21, 6, 1)$ designs fail Fisher's Inequality, and so cannot exist. All of these parameters sets are recorded in a header file, with one set being 'active.'

Table 10.1: The admissible Steiner systems, $v \leq 32$ and $b < 1000$

design	r	b	n	design	r	b	n
2-(7,3,1)	3	7	1	3-(8,4,1)	7	14	1
2-(9,3,1)	4	12	1	3-(10,4,1)	12	30	1
2-(13,3,1)	6	26	2	3-(14,4,1)	26	91	4
2-(13,4,1)	4	13	1	3-(16,4,1)	35	140	≥ 31301
2-(15,3,1)	7	35	80	3-(17,5,1)	20	68	1
2-(16,4,1)	5	20	1	3-(20,4,1)	57	285	$\geq 10^{17}$
2-(16,6,1)	3	8	0	3-(22,4,1)	70	385	≥ 21
2-(19,3,1)	9	57	$\geq 1.1 \times 10^{9}$	3-(22,6,1)	21	77	1
2-(21,3,1)	10	70	$\geq 2 \times 10^{6}$	3-(22,7,1)	14	44	0
2-(21,5,1)	5	21	1	3-(26,4,1)	100	650	≥ 1
2-(21,6,1)	4	14	0	3-(26,5,1)	50	260	≥ 1
2-(25,3,1)	12	100	$\geq 10^{14}$	3-(26,6,1)	30	130	≥ 1
2-(25,4,1)	8	50	18	3-(28,4,1)	117	819	$\geq 4 \times 10^{20}$
2-(25,5,1)	6	30	1	4-(11,5,1)	30	66	1
2-(27,3,1)	13	117	$\geq 10^{11}$	4-(15,5,1)	91	273	0
2-(28,4,1)	9	63	≥ 145	4-(17,5,1)	140	476	?
2-(31,3,1)	15	155	$\geq 6 \times 10^{16}$	4-(18,6,1)	68	204	0
2-(31,6,1)	6	31	1	4-(23,7,1)	77	253	1
				5-(12,6,1)	66	132	1
				5-(16,6,1)	273	728	0
				5-(24,8,1)	253	759	1

This set of parameters is that used by the completion routine, and any programmes that incorporate it.

The parameters are thus compile-time constants, allowing the compiler's optimiser to perform constant-folding; that is, they can be treated as constants at compile time, and expressions involving them can be partially evaluated by the compiler. All arrays are just big enough to contain their maximum data. Of course, programmes have to be recompiled if a different set of parameters is considered.

We avoid memory allocation problems for the feasible and required lists, and simplify their handling, by building these in local storage for each call to the search routine, as opposed to updating single copies of each. The arrays used are fixed size, and are allocated as part of the normal local-variable allocation from the stack during the usual function-call handling. Thus, our storage allocation and deallocation is essentially 'free,' although it is wasteful.

Remark: If $b-|S|$ and $\binom{v}{k}$ are large, the total amount of memory used by the recursive calls to the search routine may exceed the default amount of stack space. The depth of the call stack depends on $b-|S|$, while $\binom{v}{k}$ is the main factor in determining the amount of local storage for each call. If problems occur, the programme's stack allocation should be increased, or local storage for the arrays dynamically allocated at entry.

Although the number of parameter sets seems limited, note that it contains Steiner triple systems, Steiner quadruple systems, projective and affine planes, and both the small and the large Mathieu designs. The sizes and distributions of defining sets of these are of interest, and the 39 designs contain examples where these are not yet known or fully enumerated.

Finally, `complete`'s method of handling completions is to store them on a list, and return this list to the user when all completions – or up to the requested number, if some maximum number of completions has been specified – have been found. This can cause the allocation of a large amount of memory, which it is the user's responsibility to deallocate. It is also not very flexible, and can lead to long waiting times. The `comp03` and `comp04` routines are designed to 'pause' each time a new completion is found, to allow the user to process it. A user-supplied function is called at each completion, then the search is continued. The user's routine can be used to abort the search, if no more completions are required. If no user-supplied function is present, the default is simply to print out and count the completions.

10.5 Performance tests

Assessing the performance of a completion routine, and comparing one routine with another, turned out to be a challenging task. The results depend on the parameter set, which of the non-isomorphic designs is being used and the number of blocks in the partial relative to b. Interestingly, they also depend strongly on which particular partial of a given size is chosen. Thus, the results are highly case-specific, and predicting the performance is difficult.

Accordingly, in this section, we present a series of vignettes detailing the results of a succession of case studies. Each one is intended to illustrate a particular aspect of performance, and is accompanied by a brief commentary. The coverage is not exhaustive, but the examples include cases where `comp03` and `comp04` were used to obtain new results.

All test runs were on a SPARC-based machine, clocked at 60MHz. Times listed are the 'user' times as reported by the `time` command, running under the `ksh` shell. `Time` provides a resolution of 0.01 second, so times less than, say, 0.1 second should be interpreted with care, since they may not accurately reflect the actual CPU time

Table 10.2: The basic tests

	`complete`		`comp03`		`comp04`	
partial	non-opt	opt	non-opt	opt	non-opt	opt
2-(15,3,1)	8m03.35s	5m05.22s	0.084s	0.045s	0.085s	0.041s
2-(16,4,1)	12m18.97s	7m13.40s	0m4.59s	0m1.12s	0m4.67s	0m1.23s
3-(10,4,1)	0m00.52s	0m00.29s	0m2.04s	0m0.61s	0.047s	0.032s

used. If more accurate timing was necessary, a continuous series of, say, 100 runs was timed, and the average used. (These averages are rounded to the nearest 0.001 second.) Unoptimised code used no compiler switches, while optimised code was generated using the '-O2' switch. Except where otherwise noted, all testing was done on optimised versions of the programmes. To make the comparisons with `complete` as fair as possible, `complete`'s '-s' option (that is, simple) was used.

REMARK: In some of the tests, the ratio of running times for `complete` and the new completion routines is many orders of magnitude. This large ratio makes comparative testing difficult, since one of the times may be very small or very large. The test runs chosen are attempts to work within these limitations, while still illustrating the relative performance of the routines.

10.5.1 Basic tests

This set of runs is intended to give a general picture of the performance of the three completion routines, both in absolute terms and relative to one another. The changes in performance for different combinations of t and k are also examined. Finally, the effectiveness of compiler optimisation is tested. The results of the tests are given in Table 10.2.

The 2-(15,3,1) partial is a defining set for the fortieth design listed in [29]. Here, $|d_sD| = 11$, and a smallest defining set is {012, 034, 179, 1ce, 26d, 29c, 38b, 3ad, 45e, 4ab, 69b}. This defining set was found using the techniques of Section 10.6. There is only a single 2-(16,4,1) design. A random example was generated using `complete`, the blocks put in lexicographic order, and the last four used as a partial. This yielded the partial {38ae, 48bf, 57ce, 69ad}, which has 432 completions. For the 3-(10,4,1) design, the four block defining set {0123, 0456, 2579, 3678} from [53] was used.

The first thing to note is the effect of compiler optimisation. For a given partial, the speed-up obtained using the optimiser is generally better for the `comp03` and `comp04` programmes than for the `complete` programme, sometimes by as much as

Table 10.3: The tree sizes

partial	`complete`	`comp03`	`comp04`
2-(15,3,1)	415728	375	375
2-(16,4,1)	581432	14159	14159
3-(10,4,1)	258	10413	136

a factor of two. The apparent improvement in optimisation for the longer running times for `comp03` and `comp04` is due to the fact that, for small running times, most of the computation is unavoidable set-up, not backtrack search. The second point to note is that for designs where $t = 2$ there is little to choose between `comp03` and `comp04`. The slightly more complicated code required to handle $(t-1)$-subsets as opposed to 1-subsets does not have a significant speed impact.

When $t = 3$ the 1-subset counting heuristic is a poor choice, with `comp03` actually being slower than `complete`. The $(t-1)$-subset counting of `comp04` yields a speed-up over `comp03` of nineteen times. Note that the ratio of running times between `complete` and the better of `comp03`/`comp04` is very variable, being 7444, 387 and 9.1 for the three partials considered, for the optimised code.

10.5.2 Profiling information

The `prof` family of utilities allow a running programme to be profiled. We can use this to find the number of calls to the recursive search routine and thus the size of the search tree that is built. This allows us to assess how efficient our heuristics are at pruning the tree. For this series of tests the same partials as in the previous subsection were used. The results are presented in Table 10.3.

The tree size reductions are respectively 1108, 41 and 1.9. If we assume that running time is proportional to tree size, these ratios can be regarded as the potential speed-up from the intersection heuristic. Dividing these ratios into those of the running times from the last subsection we obtain 6.7, 9.4 and 4.8. These final ratios can be regarded as the ‘non-heuristic’ speed-up of `comp03` and `comp04` over `complete`. That is, more efficient programming techniques and data structures yield a speed-up of, very roughly, an order of magnitude in the time to process each node of the search tree.

Remark: Most of the running time of the programme (c. 90%) is accounted for by the recursive search routine, with the cardinality function `card` accounting for the remainder. Although `card` is very efficient, the intersection heuristic means that it is called frequently. If timing is critical, the function `card` can be changed to an inline function; that is, the compiler replaces each function-call by the body of the function, eliminating the function-call and return overheads. Inlining is not part of ANSI C, but is supported by many compilers, including `gcc`.

10.5.3 Completion rate

In the tests described so far, a single completion was performed. If a succession of completions have to be done, then some of the necessary setup – for example, building the initial list of k-subsets of V – is common to each completion, and thus its cost can be amortised. In some of the applications discussed in Section 10.6, large numbers of partials from a design are tested. It is of interest to find the 'completion rate' in such circumstances.

In an attempt to find smallest defining sets of size eleven for $STS(15)$, random collections of 12–15 blocks were taken from a design. These were then tested to see whether or not they were defining sets. If they were, an attempt was made to see whether or not they contained any smaller defining sets. This was done by removing up to four blocks from the partial, in all possible ways, and testing the resulting reduced partial to see whether or not it was a defining set.

During a series of tests on one particular design (W_{33} of Table 11.1) random partials of twelve blocks were generated, tested and 'minimised' at an average rate of 270 per second. (In fact, since $|d_s W_{33}| = 12$, all defining sets found were minimal.)

For designs W_{38} and W_{47}, 11 and 10-block subsets, respectively, were tested to see if any were defining sets. The overall average rate of subset generation and completion testing for the two designs was 246 and 224 per second respectively. Note that the larger partials yield the faster rate, as we would expect, since they require less work in general to complete.

10.5.4 No completions

The purpose of this test is to examine performance in the case where the partial has no completions. A secondary concern is to look at the performance with 'higher' values of t, v or k. The easiest way to obtain partials with no completions is to choose a parameter set for which a design does not exist. Among the parameter sets listed in Table 10.1, there are six cases where designs are known not to exist. Testing was done on the 2-(16,6,1), the 2-(21,6,1) and the 4-(18,6,1) cases. The

Table 10.4: The no completions tests

partial	`complete`	`comp03`	`comp04`
2-(16,6,1)	0m1.40s	0m0.55s	-
2-(21,6,1)	5h28m37.11s	0m5.28s	-
4-(18,6,1)	>72h51m45.12s	-	0m0.11s

results obtained are given in Table 10.4.

For the 2-(16,6,1) and 2-(21,6,1) designs, the partials used consisted of the block 012345 and the pair of blocks {012345, 06789a} respectively; note that these partials would represent an eighth and a seventh of the blocks of the designs. For the relatively small v value of 16, `complete` is slower than `comp03` by a factor of only 2.5. However, when v is increased to 21, while `comp03` slows down by a factor of 9.6, `complete` slows down by a factor of 14084, yielding a speed ratio of 3734.

For the 4-(18,6,1) design $\lambda_{t-1} = \lambda_3 = 5$, and it is easy to see that, if such a design did exist, we could assume without loss of generality that it contained the blocks {012345, 012678, 0129ab, 012cde, 012fgh}. The triple 013 must appear four more times, so we 'arbitrarily' add the blocks {01369c, 0137af, 0138dg, 013beh}. Adding the further block {0146ae} yields a 10-block partial that is processed by `comp04` in 0.11s, with a tree size of only seven nodes. `Complete` had still not finished after three days – after which it was aborted – but its running time was long enough to indicate that `comp04` is faster than it by at least six orders of magnitude.

It seems to be the case that the performance of `complete` degrades rapidly as the number of combinations $\binom{v}{k}$, $\binom{v}{t}$, $\binom{v}{t-1}$, $\binom{k}{t}$ or $\binom{k}{t-1}$; this accords with anecdotal evidence from the existing body of users of programmes based on `complete`. The intersection heuristic used in `comp04` seems to be less sensitive to these parameters.

10.5.5 Partial size

The aim of this series of tests is to investigate how the running times of the completion routines vary as the size of a partial is increased. We are also interested in how the order in which blocks are added to a partial affects these running times. The 2-(19,3,1) designs were chosen for this test, and a random example was generated using a partial-completion programme based on `comp03`. The techniques described in Section 10.6 were then used to find a random 24-block minimal defining set of this design. This partial was completed by both `complete` and `comp03`, and the orders in which they added blocks to the partial were recorded. Two series of larger and

Table 10.5: The varying partial size tests

	ordered by `complete`			ordered by `comp03`		
$\lvert dD\rvert$	`complete`	`comp03`	#calls	`complete`	`comp03`	#calls
24 (min)	3h31m58.05s	0.193s	5659	3h31m45.27s	0.196s	5659
25	18m07.78s	0.054s	624	1h19m30.69s	0.191s	5603
26	2m32.45s	0.047s	457	40m19.22s	0.188s	5570
27	1m12.56s	0.038s	289	16m17.32s	0.188s	5567
28	0m06.94s	0.036s	288	16m16.75s	0.187s	5566
29	0m05.33s	0.029s	129	6m41.93s	0.178s	5491
30	0m03.86s	0.031s	101	2m05.29s	0.178s	5488
31	0m02.44s	0.028s	98	2m05.11s	0.178s	5487
32	0m02.46s	0.030s	97	0m26.20s	0.069s	1475
33	0m01.39s	0.026s	69	0m10.41s	0.063s	1471
34	0m00.58s	0.028s	46	0m10.27s	0.069s	1470
35	0.170s	0.028s	46	0m04.21s	0.040s	503
36	0.169s	0.029s	45	0m01.26s	0.041s	458
37	0.213s	0.028s	73	0m01.26s	0.040s	457
38	0.152s	0.027s	42	0m00.60s	0.031s	184
39	0.043s	0.027s	29	0m00.23s	0.031s	87
40	0.045s	0.026s	28	0m00.25s	0.028s	86
41	0.041s	0.026s	27	0m00.11s	0.028s	32

larger defining sets were then generated by adding blocks to the minimal defining set in the same order. Both series of partials were then run through both completion routines, recording the running times and, for `comp03`, the number of recursive calls. The results are recorded in Table 10.5.

In general, the running times decrease as the partial sizes increase. However, the results depend strongly on which blocks are added to increase a partial's size. In fact, a 'bad' choice of block can even cause the running time to increase; cf. the results for the 37-block partial.

For `complete`, the difference between the two orderings is very marked. This difference is less obvious for `comp03`, but note the sudden decrease in running time (and number of calls) when the partial size is increased from 24 to 25, using `complete`'s ordering. This suggests that, in this case at least, `complete`'s heuristic for selecting a block from the required list is better that `comp03`'s.

For a given ordering, `comp03` is always faster than `complete`, but its advantage varies between the orderings and with partial size. Note the 'step-like' reductions in running time with partial size and, for `comp03`, their correlation with reductions in the size of the search tree.

The running time for `comp03` levels out when it reaches approximately 30 milliseconds, and does not reduce further. This behaviour is characteristic of completion routines. If the partial is 'large' in relation to $|dD|$ or is a 'substantial' fraction of b, completion times are not very sensitive to the size of a partial or its exact composition. In these cases the tree size is small, and most of the running time is due to the overhead of starting the programme and initialising the data structures.

This setup time puts a lower bound on the running time of the `comp03` and `comp04` routines. The main factor is the time to build the list of k and $(t-1)$-subsets of V. When a succession of runs is being performed, this cost can be amortised over the runs. However, when a single run is performed, or when $\binom{v}{k}$ becomes large, this setup time impacts the shortest attainable running time.

10.5.6 Partials of fixed size

For fixed size partials in a given design, some partials are more 'difficult' to complete than others. The aim of this section is to illustrate the timing differences among partials of a given size in a given design. For this series of tests, we again use the 3-(10,4,1) design, which has $|d_sD| = 4$. In [53] the 43 isomorphism classes of 4-block subsets of the design are listed. The representatives listed for each class were used as the input partials, and the results are given in Table 10.6. (The class used in Section 10.5.1 was class 3.) The first column is the class number i from [53], with the second column being the number of completions. The third, fourth and fifth columns are the running times for `complete`, `comp03` and `comp04`.

The results for this test are very interesting, and reinforce the comments made earlier regarding the difficulty of predicting performance. For each of the routines there is a wide variation in the time required to complete a partial, with the ratio of the maximum to minimum times being 6.5, 21.1 and 3.6 for the three routines respectively. The timings for each routine are well scattered throughout the range, and there seems to be no correlation between the timings for the three routines.

Note that `comp04` is always faster than `complete`, but the speed advantage varies from 31.8 (class 40) to 5.1 (class 9). `Comp04` is also always faster than `comp03`, with the speed advantage being 41.7 (class 18) to 3.5 (class 21). Between `complete` and `comp03`, the advantage depends on which partial is used. For class 23 `comp03` is faster than `complete` by a factor of 7.0, while for class 9 `complete` is the faster by a factor of 7.1.

Table 10.6: The partials of four blocks of the 3-(10,4,1)

class	#com	`complete`	`comp03`	`comp04`	class	#com	`complete`	`comp03`	`comp04`
1	24	0.607s	2.150s	0.090s	23	3	0.716s	0.102s	0.029s
2	24	1.563s	0.424s	0.077s	24	1	0.730s	0.280s	0.025s
3	1	0.269s	0.621s	0.033s	25	1	0.336s	0.222s	0.027s
4	4	0.624s	1.096s	0.031s	26	2	0.813s	0.321s	0.030s
5	4	0.778s	0.358s	0.035s	27	2	0.614s	0.285s	0.028s
6	6	0.848s	0.512s	0.038s	28	2	0.405s	0.223s	0.029s
7	8	0.519s	0.592s	0.049s	29	2	0.494s	0.293s	0.031s
8	8	0.387s	0.489s	0.045s	30	4	0.424s	0.364s	0.032s
9	8	0.242s	1.727s	0.047s	31	4	0.731s	0.344s	0.033s
10	6	1.093s	0.512s	0.039s	32	4	0.895s	0.406s	0.033s
11	4	0.450s	0.151s	0.039s	33	4	0.350s	0.532s	0.035s
12	4	0.688s	0.431s	0.034s	34	4	0.312s	0.438s	0.035s
13	4	0.788s	0.424s	0.033s	35	6	0.435s	0.162s	0.037s
14	4	0.542s	1.210s	0.037s	36	6	0.401s	0.418s	0.047s
15	4	0.542s	0.301s	0.034s	37	6	0.843s	0.149s	0.039s
16	4	0.538s	0.412s	0.035s	38	2	0.727s	0.306s	0.030s
17	4	0.264s	0.964s	0.032s	39	2	0.769s	0.322s	0.028s
18	4	0.292s	1.377s	0.033s	40	4	1.018s	0.421s	0.032s
19	4	0.298s	1.318s	0.039s	41	4	0.520s	0.331s	0.031s
20	6	1.240s	0.455s	0.040s	42	6	0.626s	0.169s	0.036s
21	12	0.719s	0.174s	0.050s	43	4	0.612s	0.365s	0.032s
22	12	0.425s	0.267s	0.046s					

10.6 Applications

As the behaviour of the new completion routines was investigated, and as they were used to study defining sets for a variety of designs, a range of utilities using them was developed. In the subsections that follow we discuss these applications of completion routines. The results given in Chapter 11 were obtained by utilising these techniques.

10.6.1 Completing, embedding & extending partials

The first utility developed was, of course, a simple programme which, given a partial, generates all of its completions. This is intended to test whether or not partials complete to designs and, if they do, whether or not they are defining sets. It is useful for checking putative defining sets obtained by other methods, but does not directly assist in the determination of $|d_sD|$.

The completions of an empty partial are all the distinct designs. When enumerating all the designs for a given set of parameters we normally only want one example of each non-isomorphic design. That is, when searching, we reject isomorphs of designs already found. See [68] for an algorithm for enumerating the designs with given parameters. We do not reject isomorphs when completing a partial, since it is often the case that a partial is in many distinct, but isomorphic, completions.

So although our completion utility does not enumerate designs in a very efficient manner, it is sometimes convenient to use it to produce ‘random’ designs. For example, the number of non-isomorphic 2-(19,3,1) designs is not known, but is at least 1.1×10^9 (recall Table 10.1). A completion utility based on `comp03` produces distinct 2-(19,3,1) designs at the rate of about 30 per second.

If we wish to produce example designs containing particular configurations of blocks, if these exist, we simply supply a partial consisting of the configuration to the utility. For example, the 80 2-(15,3,1) designs contain varying numbers of copies of the Pasch configuration, from 105 to none. However, if the utility is run with the two disjoint (in the sense that they have no point in common) Pasch configurations {012, 034, 513, 524, 678, 69*a*, b79, *b*8*a*} as the input partial, no completions are found. Thus, no 2-(15,3,1) system contains a pair of disjoint Pasch configurations. Running time was approximately four seconds.

A similar attempt with 2-(19,3,1) systems rapidly produces many examples which contain two disjoint Pasch configurations. In fact, there are 2-(19,3,1) designs with three disjoint Pasch configurations; this is obviously the best possible.

REMARK: It might be interesting to investigate, for a given configuration of blocks and value of c, the spectrum for Steiner triple systems that contain c disjoint copies of the configuration.

A particular instance of embedding a partial is the problem of embedding one design in another; that is, finding a design that contains a given design as a subdesign. It is known, see [24], that a $2\text{-}(v, 3, 1)$ design can be embedded in some $2\text{-}(w, 3, 1)$ design if and only if w is admissible and $w \geq 2v + 1$. For example, the completion utility produced distinct $STS(15)$ containing a copy of the Fano plane at the rate of 45 per second.

Despite this use of the completion routine, the problem of completing a partial is different from that of embedding a partial. In the former, the parameters are fixed and known, while in the latter, the parameter v can be varied. For a survey of some results on embeddings, see [83]. To find an embedding, a non-enumerative technique such as hill-climbing can be used. See, for example, [30, 107]. Such techniques could obviously be adapted to find one or two completions. Finding one completion proves that the partial completes to a design, while finding at least two proves that it is not a defining set. Such techniques cannot find all completions, or prove that none exist.

We can also use our completion routine to investigate extensions of designs. It is an open problem whether or not every $STS(v)$ can be extended to an $SQS(v+1)$. However, it is known that this is true for $v = 7, 9, 13, 15$; the three smaller cases are straightforward, while the case $v = 15$ was completed in [23]. There are two $STS(13)$ and four $SQS(14)$, with the two $STS(13)$ having 26 and 65 extensions [92]. In both cases, the completion utility based on `comp04` took ≈20 minutes to produce a complete list of the extensions. Given smallest defining sets for the $STS(13)$, these lists can be used with Lemma 2.38 to find upper bounds for $|d_sD|$ for the $SQS(14)$.

10.6.2 Finding minimal defining sets

If a random selection of s blocks from a design D completes uniquely, then $|d_sD| \leq s$. By attempting to find a defining set of a designated size and, if successful, then trying to find one of the next smaller size, we can obtain a succession of upper bounds for $|d_sD|$. Despite the crudity of this technique, it is surprisingly effective, although its efficacy depends critically on the speed of completion, and on the density of defining sets of particular sizes.

REMARK: Note that testing whether or not a set of blocks is a defining set does not mean that all completions of a partial must be found. If a partial is not a defining set, completion can be aborted as soon as a second completion is found. Only if a partial is a defining set must the completion routine be run to its conclusion.

Given a defining set, it may contain a proper subset which is also a defining set; that is, it may be non-minimal. When searching for 'small' defining sets, it is often more efficient to search for defining sets slightly larger than required and then attempt to minimise them.

When minimising defining sets, note that if a set of blocks is not a defining set, then no subset of it is a defining set. Thus, given a defining set S, we can divide the blocks in it into two classes, **essential** and **non-essential**. Any proper subset of S that is a defining set must contain all the essential blocks. It is easy to classify the blocks, and test for minimality, by trying to complete the s partials that result from deleting a single block from S.

Suppose our s-block partial $S = \{b_0, \ldots, b_{s-1}\}$. Then, if $S \setminus \{b_i\}$, for some b_i, $0 \leq i \leq s-1$, does not have a unique completion, the block b_i is essential and must be in any subset of S that is a defining set. If $S \setminus \{b_i\}$ has a unique completion, then b_i is non-essential, S is non-minimal, and we have found a smaller defining set. If $S \setminus \{b_i\}$ does not have a unique completion for all $0 \leq i \leq s-1$, then all the blocks are essential, and S is minimal.

REMARK: The technique quoted in [93, Lemma 9.17] is a faster method of testing whether or not a defining set is minimal, as it uses larger-sized partials. However, it does not yield a smaller defining set from a non-minimal defining set.

By successively removing non-essential blocks, we can find all the minimal defining sets contained in a given defining set, and thus the smallest defining sets it contains. Of course, these need not in general be smallest defining sets for the design. In practice, such an exhaustive search may not be feasible, and some limit must be put on the amount of minimisation attempted for each random partial. With a 'good' choice of partial sizes and amount of minimisation, this technique has proved to be very useful.

The opposite technique, of starting with the essential blocks from the partial and adding non-essential blocks until a defining set is obtained, is not so useful. In practice, the number of essential blocks in a defining set can be very small, even zero. Given that we have some lower bound on $|d_s D|$, the number of ways of building up the essential blocks into a partial of at least this size is too large to make testing them all feasible.

For the 80 2-(15,3,1) designs (see the next chapter) the above techniques rapidly found defining sets that were smallest. The majority of the designs required significantly less than a day of computer time, although in some cases runs over a weekend were required.

10.6.3 Generating trades in designs

An interesting application of completion routines is the generation of trades in a design. These can be put to many uses: they can be used as part of a study of trades, either in themselves or as subsets of a design; the algorithm for finding defining sets described in [51] eliminates partials that do not intersect with any of a list of trades, and thus cannot be defining sets; the BILP formed from a collection of trades in a design can be used to provides a lower bound on $|d_s D|$, as used in [71] to assist in finding smallest defining sets for the 36 2-(9,3,2) designs.

We have already described algorithms to enumerate all trades in simple designs, or all RDS in arbitrary designs. Unfortunately, the high complexity of these algorithms means that they have limited applicability. An alternative method, in the case of trades, is to note that, given a partial $S \subseteq \mathcal{B}$ which is not a defining set of D, a completion of S distinct from D defines a trade. Let such a completion be D'; then it is easy to see that $T_1 = \mathcal{B} \setminus (\mathcal{B} \cap \mathcal{B}')$ is a trade contained in D.

Thus, to generate trades in a design D, all we need do is take random partials of D, find their completions, and then compare these to D. A partial that has $c > 1$ completions defines up to $c - 1$ distinct trades in the design. Provided we have a fast completion algorithm, this technique is very effective in finding large numbers of trades. In fact, it is so effective that some method for restricting the number of trades is required. Usually, we consider only minimal trades, put a limit on the volume of the trades, or consider only some of the completions of each partial.

The random nature of the partials used means that the technique is both non-deterministic and non-enumerative. Successive runs produce different collections of trades and there is no guarantee that all (minimal) trades with less than a given volume have been generated. However, provided the sizes of the random partials are chosen carefully, good results are generally obtained.

The technique just described was the primary one used to find lower bounds for $|d_s D|$ for the 80 2-(15,3,1) designs. It is iterative, in that the trades from successive runs are accumulated until the solution to the BILP that these trades represent – that is, the lower bound on $|d_s D|$ – matches the upper bound produced by the methods of the previous subsection.

10.6.4 Lower bounds on $|d_s D|$, via exhaustion

If the list of trades available for a design is not sufficient to provide a good lower bound, an exhaustive testing technique can be used. We test all s-block subsets of $\mathcal{B}$; if none of these is a defining set, then $|d_s D| > s$.

EXAMPLE 10.2: An $STS(15)$ has 35 blocks, and has 417 225 900 11-block subsets. On a Pentium-based PC clocked at 133 MHz, it takes 10–11 days to test whether or not each of these is a defining set, with approximately 450 partials being processed each second.

Since the completion properties of a partial are invariant under isomorphisms, it might seem that it would be more efficient to divide the s-block subsets into isomorphism classes and then complete one representative of each class. This is the technique used in `bds` (see [21]), a standard utility for finding smallest defining sets and based on the algorithm described in [51]. However, to divide the subsets into classes, a programme such as `nauty` must be run on each partial to obtain its canonic form. For the 11-block partials of an $STS(15)$, it was found that completion was faster than `nauty`, so the naïve approach is faster. In other cases, it is more efficient to classify the partials first.

If the automorphism group of a design is non-trivial, we can go some way towards classifying the partials, without incurring the expense of running `nauty` on each partial, by making use of the block orbits under the action of the group. Obviously, if b_i and b_j are two distinct blocks in the same orbit, then there is an s-block defining set containing b_i if and only if there is one containing b_j. So we can pick a representative for each orbit, and need only consider partials that contain at least one of these representatives.

If the design is transitive this can be particularly effective. If not, suppose that the orbits can be partitioned into two sets $\mathcal{O}$ and $\mathcal{O}^*$ such that $| \cup_i \mathcal{O}_i^* | < s$. Then any s-block partial must contain a block from an orbit in $\mathcal{O}$. Thus we need only consider partials which contain one or more representatives of orbits in $\mathcal{O}$.

EXAMPLE 10.3: The 79th $STS(15)$ – design W_{79} of the next chapter – has four block orbits, of sizes 2, 6, 9 and 18. So, when testing 10-block subsets, to establish that $|d_s W_{79}| > 10$, only partials that contain a representative of the 9 or 18-block orbit need be tested.

10.6.5 Distribution of defining sets

As well as finding the size of smallest defining sets, we would like to enumerate all smallest defining sets. In fact, given a design, we would ideally like to classify all subsets of the design in terms of their defining properties. It is a simple matter to enumerate all partials in a given design and test them. Obviously, such exhaustive testing is only possible for the smallest designs.

Where such testing is feasible, it can be used to prove the existence or non-existence of (minimal) defining sets of a particular size, the number of these, and the number, size and automorphism groups of their isomorphism classes. Some examples are given in the next chapter.

Where such testing is not feasible, testing random subsets of the blocks of a design for their defining properties can be helpful in building up a picture of the distribution of defining sets. Of course, the results are only probabilistic, but they are interesting nonetheless. We give an example of this technique in the next chapter, where we investigate $\text{spec}_m(D)$, the spectrum of minimal defining sets, for the 80 $STS(15)$.

10.7 Conclusions

The standard heuristic to speedup the completion of partials involves keeping track of the u-subsets of V that have appeared so far in the partial, for one or more values of u, $1 \leq u \leq t$. This information allows a tree of potential solutions to be built and traversed efficiently. The intersection heuristic allows t-subset occurrence to be controlled implicitly, simply by monitoring block intersection sizes. As the performance tests show, this is very effective, and we present some of our results in the next chapter. Unfortunately, this heuristic is limited to the case $\lambda = 1$.

Monitoring block intersection sizes will not count t-subsets in the general case. However, if information is available about a design's block intersection properties, it can be a very effective heuristic for controlling feasible lists. The `complete` routine has a switch to allow it to cope with linked designs, where any pair of distinct blocks has a fixed intersection size. This could be generalised to affine or quasi-symmetric designs, where there are two possible intersection sizes, or to any case where all possible block intersection sizes are known for a particular parameter set.

The intersection heuristic is effectively a very fast method of counting t-subsets. In general the key to writing fast completion routines seems to be to keep track of the multiplicities of all u-subsets, $1 \leq u \leq t$, and then to use this information to select the 'best' next block to add to the partial. There is still considerable scope for investigating how best to do this, as the most effective method seems to be case dependent [82].

As the applications listed in Section 10.6 indicate, completion routines are very useful in investigating defining sets. However, their utility depends critically on completions being performed rapidly. So it is important that we try to identify additional properties of designs that can be used to build efficient completion routines for other sub-classes of designs.

Chapter 11

Results on Steiner designs

The techniques discussed in the previous chapter were applied to a range of Steiner systems, and a selection of the results obtained are given in this chapter. These results are of several types: values for $|d_s D|$; enumerations of smallest defining sets; results on $\text{spec}_m(D)$; and enumerations of minimal defining sets.

11.1 Smallest defining sets of the $STS(15)$

For $v = 7$, 9 and 13, there are one, one, and two non-isomorphic $STS(v)$ respectively. The sizes of smallest defining sets for these were determined in [47, 46, 51], being three, four, and eight/nine respectively. There are 80 non-isomorphic $STS(15)$ and we follow [93] in using the standard listing as given in [88], and labelling them W_1 to W_{80}. Our results are summarised in Table 11.1. The first two columns are the design's label and automorphism group order.

Recall that a Pasch configuration is a set of four blocks isomorphic to {012, 034, 513, 524}, and is a (6,3,2) trade. The number of Pasch configurations, #P, in each design is given in Table 11.1. Any defining set of W_i must intersect all Pasch configurations in W_i, and we can use these to obtain a lower bound on $|d_s(W_i)|$ by formulating and solving a BILP. These BILPs were solved using `opbdp`, and the results, LB, are given in Table 11.1.

The final column, s, in Table 11.1 gives the size of a smallest defining set. Note that the bound LB is tight in six cases (it was claimed erroneously in [98] that this was never the case) and is too low by one in a further ten cases. In Tables G.1–G.8 of Appendix G the blocks of the designs are listed. The points $\{1, \ldots, 15\}$ of [93] have been relabelled $\{\mathtt{0}, \ldots, \mathtt{9}, \mathtt{a}, \ldots, \mathtt{e}\}$. For each design the blocks of a smallest defining set are marked with a '`*`'. These smallest defining sets were obtained by the random sampling and minimising method discussed, so the particular defining set given has no significance.

TABLE 11.1: Summary of results for the $STS(15)$

D	$\lvert A\rvert$	#P	LB	s	D	$\lvert A\rvert$	#P	LB	s
W_1	20160	105	15	16	W_{41}	1	12	7	11
W_2	192	73	14	14	W_{42}	2	8	3	11
W_3	96	57	14	14	W_{43}	6	10	5	11
W_4	8	49	12	13	W_{44}	2	8	5	11
W_5	32	49	12	13	W_{45}	1	9	4	11
W_6	24	37	11	12	W_{46}	1	7	4	11
W_7	288	33	11	13	W_{47}	1	10	5	11
W_8	4	37	12	12	W_{48}	1	8	4	11
W_9	2	31	11	12	W_{49}	1	7	4	11
W_{10}	2	31	11	12	W_{50}	1	6	4	11
W_{11}	2	23	9	11	W_{51}	1	9	4	11
W_{12}	3	32	10	12	W_{52}	1	9	4	12
W_{13}	8	33	11	12	W_{53}	1	10	5	11
W_{14}	12	37	12	13	W_{54}	1	11	5	11
W_{15}	4	25	11	11	W_{55}	1	9	4	11
W_{16}	168	49	14	14	W_{56}	1	8	4	11
W_{17}	24	25	11	12	W_{57}	1	5	4	11
W_{18}	4	25	11	11	W_{58}	1	8	5	11
W_{19}	12	17	9	11	W_{59}	3	13	6	12
W_{20}	3	20	9	11	W_{60}	1	7	4	11
W_{21}	3	20	7	11	W_{61}	21	14	7	12
W_{22}	3	17	5	11	W_{62}	3	7	4	11
W_{23}	1	18	8	11	W_{63}	3	7	6	11
W_{24}	1	19	8	12	W_{64}	3	10	7	12
W_{25}	1	20	10	11	W_{65}	1	7	3	11
W_{26}	1	23	10	12	W_{66}	1	6	3	11
W_{27}	1	14	8	11	W_{67}	1	5	3	11
W_{28}	1	15	8	11	W_{68}	1	6	3	11
W_{29}	3	19	10	12	W_{69}	1	5	3	11
W_{30}	2	14	7	11	W_{70}	1	9	6	11
W_{31}	4	18	8	11	W_{71}	1	5	3	11
W_{32}	1	13	7	11	W_{72}	1	5	2	11
W_{33}	1	12	7	12	W_{73}	4	6	2	11
W_{34}	1	12	7	12	W_{74}	4	8	4	11
W_{35}	3	13	7	12	W_{75}	3	7	4	11
W_{36}	4	10	6	11	W_{76}	5	10	8	12
W_{37}	12	6	3	11	W_{77}	3	2	1	11
W_{38}	1	9	4	12	W_{78}	4	6	4	11
W_{39}	1	12	6	11	W_{79}	36	6	2	11
W_{40}	1	13	7	11	W_{80}	60	0	0	12

For 63 of the 80 values of $|d_sW_i|$, the lower bound was proved using a collection of trades. For the remaining 17, the trade-based lower bound and the upper bound differed by one. In each case, an exhaustive completion-testing of partials showed that no partial of the smaller size completed uniquely.

In [93] the nine values of $|d_sW_i|$ for $i \in \{1, \ldots, 8, 16\}$ were determined. Bounds were given for all the other W_i; of these, 4 lower bounds and 40 upper bounds are tight. Note that neither $|A|$ nor #P is monotonic with $|d_sW_i|$. However, the coefficient of linear correlation between $|d_sW_i|$ and #P is +0.766. The design W_1 arises from the lines of the geometry $PG(3,2)$, and contains the maximum possible number of Pasch configurations; that is, $v(v-1)(v-3)/24$, see [108]. If we regard W_1 as atypical, and omit its data, the coefficient becomes +0.800.

11.2 Minimal defining sets of the $STS(15)$

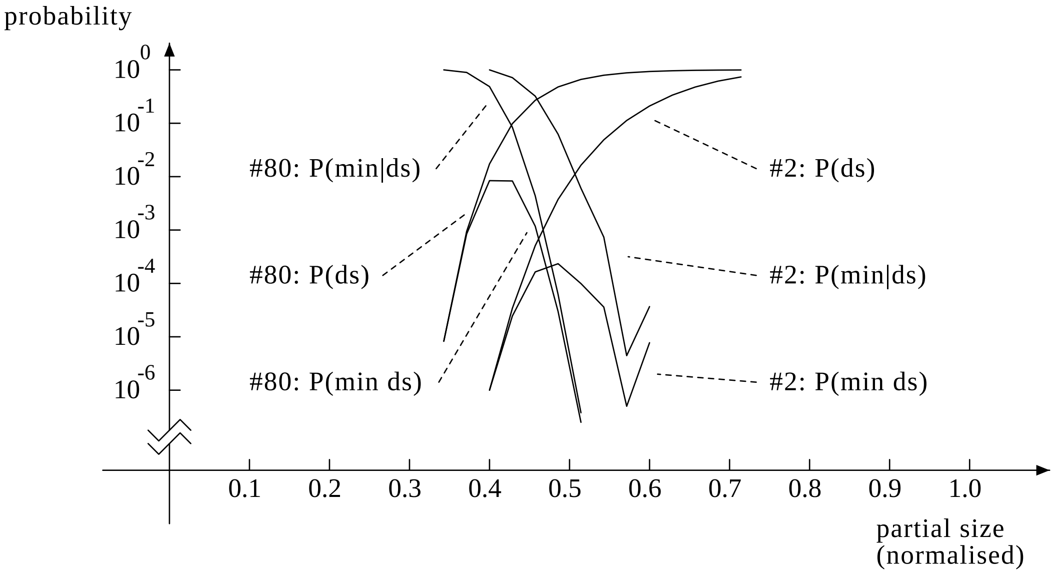
Recall the definition of $\text{spec}_m(D)$ in Section 2.3. The `comp03` completion utility is fast enough to allow a large number of partials from each $STS(15)$, for each of a range of partial sizes, to be tested for their completion properties. Those partials that are defining sets are counted, and then tested for minimality. For each partial size of each design, a total of 4 000 000 'random' partials was generated and tested; total running time was 102 days. The `rand` function was used to select blocks of the design uniformly at random. Although `rand` is not truly random, being a multiplicative congruential generator, its period of 2^{32} is very much longer than the number of values required during each run of 4 000 000 partials. Seeding the generator with the time of day before each run ensures that the runs are independent.

The results of these tests are given in Tables G.9–G.24. The left hand column in these tables is the partial size, and for each design we list the number of defining sets and minimal defining sets encountered. A '-' indicates that the partial size is less than $|d_sW_i|$ for this design. Note that results are probabilistic; only $\text{spec}_m(W_1)$ and the 80 values of $|d_sW_i|$ have been proved. It is likely that larger samples would extend some of the spectra slightly. So the discussion in the following paragraphs should be read with these caveats in mind.

The design derived from the geometry, W_1, has the largest value of $\max(\text{spec}_m(W_i))$ of all the designs, at 22. One and three of the designs have values of 21 and 19 for $\max(\text{spec}_m(W_i))$ respectively, the remainder have values of 18 or 17. None of the designs have holes in their spectra. All the designs, apart from W_1 and W_2, have a 'smooth' distribution of minimal defining sets. The 'spike' in $\text{spec}_m(W_2)$ is particularly marked; the reason for this is unknown.

The size of the spectrum, $|\text{spec}_m(W_i)|$, varies from 5 to 8, with there being 2, 9, 48 and 21 designs with each size, respectively. The general shape of the $\text{spec}_m(W_i)$, for $3 \leq i \leq 80$, is very similar. However, the probability that a random partial of a particular size is a minimal defining set, calculated at the 'peak' of the distribution, is very variable, ranging from 0.000234 for W_2 to 0.008414 for W_{80}.

Figure 11.1: The distribution of defining sets in W_2 and W_{80}



Note that smallest defining sets are 'rare,' with only 26 of the 80 designs yielding sample ones. However, in 30 of the remaining cases a non-minimal defining set of size $|d_sW_i| + 1$ was found, while a non-minimal defining set of size $|d_sW_i| + 2$ was found in all remaining cases. This supports the claim in Subsection 10.6.2 that the random search and minimisation technique discussed is an efficient method for finding small defining sets.

Note that all designs other than W_{80} contain trades of volume 4 and thus any collection of 32 or more blocks must be a defining set, and no collection of 33 or more blocks can be a minimal defining set. The smallest trades in W_{80} have volume 6, so the corresponding limits here are 30 and 31 blocks. Note that, of the 52 360 31-block partials of a design, the number which are not defining sets of W_i is equal to the number of Pasch configurations in W_i.

To compare data on the distribution of defining sets for several designs, it can be helpful to graph the information obtained from samples of partials. This is illustrated in Figure 11.1 for the two $STS(15)$ W_2 and W_{80} – labelled #2 and #80. Recall that these are the designs with the minimum and maximum peak probabilities that a partial is a minimal defining set. For each partial size we calculate $P(\text{ds})$, $P(\text{min ds})$, and $P(\text{min}|\text{ds})$; these are respectively, the probability that a random partial of the given size is a defining set, a minimal defining set, or a minimal defining set given that it is a defining set. We plot the probabilities vertically, using a logarithmic scale. The size of the partials, s, is normalised to s/b.

Figure 11.2: The 3-(17,5,1) inversive plane I

0)	0	1	2	3	4	1)	0	1	5	6	7	2)	0	1	8	9	10	3)	0	1	11	12	13
4)	0	1	14	15	16	5)	0	2	5	8	11	6)	0	2	6	9	14	7)	0	2	7	12	15
8)	0	2	10	13	16	9)	0	3	5	10	15	10)	0	3	6	11	16	11)	0	3	7	9	13
12)	0	3	8	12	14	13)	0	4	5	13	14	14)	0	4	6	10	12	15)	0	4	7	8	16
16)	0	4	9	11	15	17)	0	5	9	12	16	18)	0	6	8	13	15	19)	0	7	10	11	14
20)	1	2	5	9	13	21)	1	2	6	12	16	22)	1	2	7	8	14	23)	1	2	10	11	15
24)	1	3	5	11	14	25)	1	3	6	9	15	26)	1	3	7	10	12	27)	1	3	8	13	16
28)	1	4	5	10	16	29)	1	4	6	8	11	30)	1	4	7	13	15	31)	1	4	9	12	14
32)	1	5	8	12	15	33)	1	6	10	13	14	34)	1	7	9	11	16	35)	2	3	5	7	16
36)	2	3	6	8	10	37)	2	3	9	11	12	38)	2	3	13	14	15	39)	2	4	5	6	15
40)	2	4	7	9	10	41)	2	4	8	12	13	42)	2	4	11	14	16	43)	2	5	10	12	14
44)	2	6	7	11	13	45)	2	8	9	15	16	46)	3	4	5	8	9	47)	3	4	6	7	14
48)	3	4	10	11	13	49)	3	4	12	15	16	50)	3	5	6	12	13	51)	3	7	8	11	15
52)	3	9	10	14	16	53)	4	5	7	11	12	54)	4	6	9	13	16	55)	4	8	10	14	15
56)	5	6	8	14	16	57)	5	6	9	10	11	58)	5	7	8	10	13	59)	5	7	9	14	15
60)	5	11	13	15	16	61)	6	7	8	9	12	62)	6	7	10	15	16	63)	6	11	12	14	15
64)	7	12	13	14	16	65)	8	9	11	13	14	66)	8	10	11	12	16	67)	9	10	12	13	15

11.3 The $3\text{-}(17, 5, 1)$ inversive plane

The unique 2-(16,4,1) design is the affine plane of order 4. This design has 6 distinct extensions to a 3-(17,5,1) design, all of them isomorphic. So, since the restrictions of any 3-(17,5,1) design are 2-(16,4,1) designs, the 3-(17,5,1) design is unique. The 3-(17,5,1) design is an inversive plane, and we use the example given in Figure 11.2. The blocks containing 0 are the extended affine plane, and the extension is the first in lexicographic order.

We label the design of Figure 11.2 I, and its 68 blocks $0 \ldots 67$. Now the affine plane has $|d_s D| = 7$, and each of its six extensions contains a block not in any of the others. So $|d_s I| \leq 8$ by Lemma 2.38, while the lower bound from Lemma 2.20 is 6. We use `comp04` and the techniques of the previous chapter to show that $|d_s I|$ is, in fact, 7.

Now $|\mathrm{aut}(I)| = 16320$, and I is transitive. So we need only test all isomorphism classes of 6 or 7 blocks from I that contain block 0, and at least 16 points (Lemma 2.20). There are 3 408 498 such sets of 6 blocks, falling into 2 298 isomorphism classes, and none of these is a defining set. There are 7 397 880 such sets of 7 blocks, falling into 36 065 isomorphism classes, and 4 439 of these classes are defining sets. Of these, 4 368 have trivial automorphism groups and 71 have groups of order 2; so the total number of smallest defining sets is

$$4368 \cdot \frac{16320}{1} + 71 \cdot \frac{16320}{2} = 71865120.$$

Lemma 11.1: *The unique* $3\text{-}(17, 5, 1)$ *design has a smallest defining set size of 7 blocks. There are* 4 368 *classes of smallest defining set with trivial automorphism*

FIGURE 11.3: Some smallest defining sets of I

order 1	order 2
0 1 2 5 6 21 27	0 1 2 5 6 27 41
0 1 2 5 6 21 50	0 1 2 5 6 50 54
0 1 2 5 6 27 30	0 1 2 5 31 39 42
0 1 8 34 43 46 55	0 1 8 17 31 62 63
0 1 8 34 46 55 60	0 1 8 18 31 34 66
0 1 8 34 46 55 67	0 1 8 32 51 54 6

group, and 71 classes with order 2. So the design has a total of 71 865 120 *smallest defining sets.* □

In Figure 11.3 we give representatives of several of the classes of smallest defining sets. The representative of a class is the lexicographically least member of the class. We include representatives of the first and last three classes in order. Interestingly, although the design is not 2-transitive, all 4 439 representatives includes blocks 0 and 1.

For I, running `nauty` on a partial is faster than running `comp04`, so the partials were grouped into classes and then tested. `Comp04` is not fast enough to test a very large number of random partials to enable an accurate picture of the distribution of defining sets in I to be built up. However, a sample of 1839 8-block partials produced 624 defining sets, of which 96 were minimal. Similar-sized samples for partial sizes 9–12 did not produce any minimal defining sets and, for the 12-block partials, 1805 out of 1899 (that is, 95%) were defining sets. Of course, since I contains trades of volume 30 (see Section 7.5), there are partials of 38 blocks that are not defining sets. By Lemma 4.44, any trade in I must have volume at least fifteen, so any set of 54 or more blocks from I completes uniquely.

11.4 A survey of small cases

Using our fast completion routine, it is possible to test all subsets of some of the smaller Steiner systems – so we can completely classify all subsets of these designs in terms of their defining properties. In this section we present some results obtained by this technique. Apart from the $STS(13)$, all the designs considered are affine or projective planes, or their extensions. So the sizes of their smallest defining sets are related by the various results reviewed in Section 2.3.

2-(7,3,1): This is the unique projective plane of order 2 – the Fano plane – and has automorphism group order 168. The only minimal, and hence the smallest, defining

Figure 11.4: The blocks of the $STS(13)$

D_1				D_2			
label	block	label	block	label	block	label	block
0	014	13	028	0	01c	13	028
1	125	14	139	1	125	14	139
2	236	15	24a	2	236	15	24a
3	347	16	35b	3	3c7	16	35b
4	458	17	46c	4	458	17	46c
5	569	18	570	5	569	18	570
6	67a	19	681	6	67a	19	681
7	78b	20	792	7	78b	20	792
8	89c	21	8a3	8	89c	21	8a3
9	9a0	22	9b4	9	9a0	22	9b4
10	ab1	23	ac5	10	ab1	23	ac5
11	bc2	24	b06	11	bc2	24	b06
12	c03	25	c17	12	403	25	417

sets consist of sets of three blocks that do not contain a common point [47]. There are precisely 28 such smallest defining sets, all isomorphic. The only minimal trades are seven copies of the Pasch trade, and all sets of four or more blocks from the design complete uniquely.

2-(9,3,1): This is the unique affine plane of order 3, and has automorphism group order 432. Smallest defining sets of this design have four blocks and consist of two blocks from each of two resolution classes [46]. Such a set of four blocks has automorphism group order 8, and occurs 54 times in the design. The only other minimal defining set has five blocks, consisting of two blocks from one resolution class and one block from each of the other three classes, with these three blocks not sharing a common point. Such a set of blocks has automorphism group order 2, and occurs 216 times in the design. The only minimal trades are 36 copies of the volume six trade T^c of Section 3.9, and all sets of seven or more blocks from the design complete uniquely.

2-(13,3,1): There are precisely two non-isomorphic $STS(13)$. Their smallest defining sets were enumerated in [53, 51]. We take the designs, labelled D_1 and D_2, from these, with the points $\{1,\ldots,9,a,\ldots,d\}$ relabelled $\{\mathtt{0},\ldots,\mathtt{9},\mathtt{a},\ldots,\mathtt{c}\}$. D_1 is cyclic, with $|\mathrm{aut}(D_1)| = 39$ and $|d_sD_1| = 9$, and D_2 is non-cyclic, with $|\mathrm{aut}(D_2)| = 6$ and $|d_sD_1| = 8$. The blocks of the two designs, with the block labelling used, are given in Figure 11.4. Note that $D_1 - D_2$ is a Pasch trade. D_1 contains 13 Pasch configurations, and D_2 has 8.

Table 11.2: The minimal defining sets of D_1

size	$\|A\|$	#classes	example	total
9	1	17	0 1 2 3 5 7 8 10 15	663
10	1	1146	0 1 2 3 4 5 6 7 8 14	44694
	3	5	0 1 2 6 8 11 16 18 21 22	65
11	1	1747	0 1 2 3 4 5 6 7 8 9 20	68133
	3	3	0 1 4 6 13 17 20 21 23 24 25	39
12	1	474	0 1 2 3 4 5 6 8 9 14 18 19	18486
	3	3	0 1 2 7 10 11 13 17 18 19 21 24	39
13	1	13	0 1 2 3 4 5 8 14 18 19 20 22 25	507
	3	1	0 1 10 13 15 16 17 18 20 21 22 23 24	13

Table 11.3: The minimal defining sets of D_2

size	$\|A\|$	#classes	example	total
8	1	2	0 2 6 8 14 15 22 23	12
9	1	864	0 1 2 3 4 5 6 7 15	5184
	2	23	0 1 2 5 6 8 12 14 15	69
10	1	15597	0 1 2 3 4 5 6 7 9 21	93582
	2	99	0 1 2 3 6 7 8 14 15 21	297
	6	2	0 3 5 13 15 19 20 21 23 25	2
11	1	14019	0 1 2 3 4 5 6 7 12 13 16	84114
	2	61	0 1 2 3 6 8 10 12 13 18 21	183
	3	1	2 4 5 9 12 16 19 20 22 23 24	2
12	1	1051	0 1 2 3 4 5 7 8 9 13 18 19	6306
	2	5	0 1 2 3 5 8 13 15 17 19 21 25	15
13	1	3	0 1 2 3 6 10 11 15 17 19 23 24 25	18

The $STS(13)$ are small enough for a complete enumeration of all minimal defining sets to be feasible. The results are presented in Tables 11.2 and 11.3. For each size where there are minimal defining sets, we list the possible automorphism groups orders, the number of classes of each order, an example, and the total number of minimal defining sets of this type.

2-(13,4,1): This is the unique projective plane of order 3, and has automorphism group order 5616. The example we take is developed from the started block 0139, and the blocks are labelled $0, \ldots, 12$, in the order they are developed. A smallest defining set has 6 blocks [46].

There are two classes of smallest defining sets [52]: one having representative {0 1 2 3 4 5} and automorphism group order 24; and one having representative {0 1 2 3 4 6} and group order 6. So the design has a total of 1 170 smallest defining sets. Exhaustive testing established that the only minimal defining sets are smallest.

2-(16,4,1): This is the unique affine plane of order 4, and has automorphism group

Figure 11.5: The blocks of the 2-(16, 4, 1)

label	block	label	block	label	block	label	block
0	`0123`	5	`147a`	10	`25af`	15	`367f`
1	`0456`	6	`158d`	11	`268c`	16	`38ae`
2	`0789`	7	`16be`	12	`27bd`	17	`48bf`
3	`0abc`	8	`19cf`	13	`34cd`	18	`57ce`
4	`0def`	9	`249e`	14	`359b`	19	`69ad`

Table 11.4: The minimal defining sets of the 2-(16, 4, 1)

size	$\|A\|$	#classes	example	total
7	2	1	0 1 2 7 8 10 17	2880
	4	1	0 1 2 7 8 10 16	1440
	6	1	0 1 2 5 7 11 12	960
	12	1	0 1 2 7 8 10 12	480
8	1	3	0 1 2 3 5 6 7 11	5760
			0 1 2 3 5 6 9 12	5760
			0 1 2 3 5 6 10 12	5760
	2	3	0 1 2 3 5 6 7 9	2880
			0 1 2 3 5 6 7 10	2880
			0 1 2 3 5 6 9 11	2880
	4	1	0 1 2 3 6 7 12 18	1440
	6	1	0 1 2 5 6 9 11 14	960

Table 11.5: The minimal defining sets of the 2-(21, 5, 1)

size	$\|A\|$	#classes	example	total
8	48	1	0 1 2 3 4 7 8 12	2520
9	2	1	0 1 2 3 4 5 6 7 9	60480
	4	1	0 1 2 3 4 5 6 7 8	30240
	6	1	0 1 2 3 4 5 6 11 12	20160
	12	1	0 1 2 3 4 5 6 8 10	10080

order 5760. A smallest defining set has 7 blocks [46], and all smallest defining sets were given [52]. We take the design given in Figure 11.5 as our example, and all minimal defining sets are enumerated in Table 11.4. Note that all representatives have blocks 0, 1 and 2 in common.

2-(21,5,1): This is the unique projective plane of order 4, and has automorphism group order 120 960. The example we take is developed from the starter block $014eg$, and the blocks are labelled $0, \dots, 21$, in the order they are developed. The smallest defining set of eight blocks given in [46] is unique, and the only other minimal defining sets are four classes of nine blocks. The results are summarised in Table 11.5. Note that all representatives have blocks 0–4 in common.

By our results on Steiner trades, a Steiner $(5, 2)$ trade has volume at least eight, and trades of volume eight or ten have a unique structure. The 2-$(21, 5, 1)$ design cannot contain any trades of volume ten, since these have foundation 22. However, it does contain $\binom{21}{2} = 210$ copies of the volume eight trade, generated by the single transpositions in S_{21}. The BILP formed from these trades has an optimal solution of eight blocks and, at least in the examples tested, is a smallest defining set.

This design has too large a v value for our trade enumeration algorithm, but some partial enumerations were undertaken. The design contains: foundation 18 trades of volume 8; foundation 19 trades of volume 12; foundation 20 trades of volumes 12 and 14; and foundation 21 trades of volumes 12 and 14–21.

2-(25,5,1): This is the unique affine plane of order 5, and has automorphism group order 12 000. A smallest defining set has 10 blocks [44]. A sampling technique was adopted for this design: 2427 out of 77349 samples of ten blocks were smallest defining sets; 23032 out of 77441 samples of eleven blocks were defining sets, and of these, 8929 were minimal; similar sized samples for 12–15 block partials failed to find any other minimal defining sets. Since Steiner $(5, 2)$ trades have volume at least eight, all partials of 23 or more blocks complete uniquely.

This design has too large a v value for our trade enumeration algorithm, but some partial enumerations were undertaken. The design contains: foundation 22 trades of volume 10; foundation 24 trades of volume 16; and foundation 25 trades of volumes 16, 18, 20 and 22–30.

2-(31,6,1): This is the unique projective plane of order 5, and has automorphism group order 370 000. A smallest defining set has 11 blocks [44]. A sampling technique was adopted for this design: 267 out of 17614 samples of eleven blocks were smallest defining sets; 4714 out of 17568 samples of twelve blocks were defining sets, and of these, 2535 were minimal; similar sized samples for 13–21 block partials failed to find any other minimal defining sets. Since Steiner $(6, 2)$ trades have volume at least ten, all partials of 22 or more blocks complete uniquely.

3-(8,4,1): The unique 3-$(8, 4, 1)$ design is the extension by complementation of the Fano plane, has automorphism group order 1344, and has a smallest defining set of three blocks [47]. In fact, the only minimal defining sets are the smallest defining sets. There is a single class of smallest defining sets, consisting of sets of three blocks with a common point, but no common pair. These are extensions of the smallest defining sets of the Fano plane, and so the class has $8 \times 28 = 224$ members.

FIGURE 11.6: The blocks of the 3-$(10, 4, 1)$

label	block	label	block	label	block
0	0123	12	2347	21	5689
1	0456	13	1358	22	4679
2	0789	14	1269	23	4578
3	0147	15	1567	24	2389
4	0258	16	2468	25	1379
5	0369	17	3459	26	1278
6	0159	18	1489	27	2356
7	0267	19	2579	28	1346
8	0348	20	3678	29	1245
9	0168				
10	0249				
11	0357				

3-(10,4,1): The 3-$(10, 4, 1)$ design is unique, has automorphism group order 1440, and is an extension of the 2-$(9, 3, 1)$ design. The design has $|d_sD| = 4$, and there are three isomorphism classes of smallest defining sets [52]. The example design we use is given in Figure 11.6. The automorphism group is transitive, so we need only test partials that contain a nominated block, say 0. All partials of eight or less blocks were tested, and the results are given in Table 11.6. Note that there are no minimal defining sets of seven or eight blocks. If $\text{spec}_m(D)$ is hole-free, then Table 11.6 lists all the minimal defining sets.

When extending the 2-$(9, 3, 1)$, the set of blocks added is a 2-$(9, 4, 3)$ design. There are three possible choices for this, and these are mutually disjoint [52]. Since a 2-$(9, 4, 3)$ design has eighteen blocks, the 54 (resp. 216) minimal defining sets of 4 (resp. 5) blocks for the 2-$(9, 3, 1)$ design give rise to $54.18 = 972$ (resp. $216.18 = 3888$) defining sets of 5 (resp. 6) blocks for the 3-$(10, 4, 1)$ design. These defining sets have a point common to 4 out of 5 (resp. 5 out of 6) of the blocks. Since the minimal defining sets of the 2-$(9, 3, 1)$ have maximum point multiplicity 2 (resp. 3), then the point on which to restrict in the 3-$(10, 4, 1)$ defining sets obtained in this way is uniquely determined. Thus such a set of defining sets exist for each point, so there are a total of 9720 (resp. 38880) defining sets in the 3-$(10, 4, 1)$ design which are extensions of minimal defining sets of the 2-$(9, 3, 1)$. Of course, these need not be minimal, but the last four example minimal defining sets in Table 11.6 do arise in this way. In total, 8 (resp. 4) of the classes of minimal defining sets of 5 (resp. 6) blocks of the 3-$(10, 4, 1)$ design arise in this way.

4-(11,5,1): The 4-$(11, 5, 1)$ design is unique, has a transitive automorphism group

Table 11.6: The minimal defining sets of the 3-$(10, 4, 1)$

size	$\lvert A\rvert$	#classes	example	total
4	2	2	0 1 3 13	720
			0 1 3 19	720
	24	1	0 1 18 19	60
5	1	15	0 1 3 22 28	21600
	2	7	0 1 3 4 21	5040
	8	1	0 1 3 4 29	180
6	1	4	0 1 3 6 11 21	5760
	2	2	0 1 3 6 10 29	1440

Figure 11.7: The blocks of the 4-$(11, 5, 1)$

label	block	label	block	label	block	label	block
30	02359	39	02378	48	12368	57	14579
31	04678	40	02457	49	12349	58	25678
32	01367	41	01389	50	12357	59	34689
33	04589	42	02689	51	24569	60	13478
34	01248	43	01345	52	34567	61	12589
35	05679	44	03479	53	14568	62	23679
36	03568	45	01256	54	35789	63	12467
37	01279	46	01578	55	16789	64	23458
38	01469	47	02346	56	24789	65	13569

Table 11.7: The smallest defining sets of the 4-$(11, 5, 1)$

size	$\lvert A\rvert$	#classes	example	total
5	1	194	0 1 3 13 30	1536480
	2	21	0 1 3 13 37	83160
	10	1	0 1 18 42 64	792

of order 7920 (the small Mathieu group, M_{11}), and is an extension of the 3-$(10, 4, 1)$ design. The 3-$(10, 4, 1)$ design has two extensions to a 4-$(11, 5, 1)$, with the added blocks of the extensions being disjoint. The design has $|d_s D| = 5$, and the 1 500 smallest defining sets of four blocks of the 3-$(10, 4, 1)$ give rise to 54 000 smallest defining sets [52].

There are, however, many more smallest defining sets. We use the example design M of [52], with blocks 0–29 being the design of Figure 11.6 extended by the point 10, and blocks 30–65 as in Figure 11.7. All sets of five blocks containing the block 0 were tested, and the results are given in Table 11.7. Although the six-block subsets were not exhaustively tested, M contains at least 865 classes of minimal defining sets of six blocks; for example, {0,1,3,4,21,32} and {0,1,3,22,26,43}, having group orders 1 and 2 respectively.

5-(12,6,1): The 5-$(12, 6, 1)$ design is unique, and has a transitive automorphism group of order 95 040 (the small Mathieu group, M_{12}). The design is the unique extension by complementation of the 4-$(11, 5, 1)$ design, and so is self-complementary and has $|d_s D| = 5$. Since the design is self-complementary, a smallest defining set cannot contain a complementary pair of blocks, as then it would not be smallest. Thus any block in a smallest defining set can be replaced by its complement without destroying the unique completion property. So each smallest defining set of five blocks of the 4-$(11, 5, 1)$ design yields $2^5 = 32$ smallest defining sets of the 5-$(12, 5, 1)$; cf. Lemma 2.37. So the 5-$(12, 5, 1)$ design has precisely $32(1536480 + 83160 + 792) = 51853824$ smallest defining sets.

APPENDIX A

Reference tables

TABLE A.1: Symbols & notation used (I)

symbol	description	reference
$\sqcap, \cap, \cup, \setminus, =, \subseteq$	multiset binary operations/relations	§8.1
$\delta(G)$	the minimum degree of G	p. 171
λ	multiplicity of t-subset in a design	p. 1
λ_0, λ_1, λ_t	the parameters b, r, λ respectively	p. 9
λ_u	multiplicity of u-subset in a design, $0 \leq u \leq t$	p. 9
λ_*	the minimum admissible λ	p. 11
λ^*	$\binom{v-t}{k-t}$, the λ for a full design	p. 11
μ	$\lvert d_sD \rvert / b$	p. 18
$\nu(G)$	the number of vertices in G	p. 171
$\triangle B$	the set of differences generated by a block	p. 10
$\triangle \mathcal{F}$	a difference set / family	p. 10
$AG(n, q)$	affine geometry, dimension n, over field of order q	p. 13
$\mathrm{aut}(D)$	automorphism group of D	p. 10
b	the number of blocks in a design	p. 1
b^*	the support size a design	p. 1
$\mathcal{B}$	the (multi-)set of blocks of a design	p. 1
$BIG(D)$	block intersection graph of a design	p. 64
cD	a class defining set of D	p. 82
c_mD	a minimal class defining set of D	p. 82
c_sD	a smallest class defining set of D	p. 82
D	a generic design, with $D = (V, \mathcal{B})$	p. 1
D_i	representative of the ith class of designs	p. 10
dD	a defining set of D	p. 3
d_mD	a minimal defining set of D	p. 3
d_sD	a smallest defining set of D	p. 3
$\lvert d_sD \rvert$, etc.	size of a smallest defining set of D, etc.	p. 3
$DS(D_1, D_2)$	discriminating set for D_1 and D_2	p. 99
D_i^l	set of disallowed volumes associated with s_i^l	p. 38
$\mathcal{D}$	transversal of the n different designs	p. 10
$\mathcal{D}^*$	set of all distinct designs	p. 10
D_i^*	set of all distinct designs isomorphic to D_i	p. 10
f (designs)	$\binom{v}{k}/b$	p. 83
f (trades)	$f(T)$, a trade's foundation size	-
$f(T)$	foundation size, of the trade T	p. 14
$F(T)$	foundation set, of the trade T	p. 14
F_t	set of allowed volumes	p. 39
$\overline{F}_t$	set of disallowed volumes	p. 39
G (graph)	a generic (simple) graph	p. 171

Table A.2: Symbols & notation used (II)

symbol	description	reference
G (group)	a generic group	p. 13
$GF[q]$	the Galois field of order q	p. 13
k	number of points in a block	p. 1
(k,t) trade	trade with no restiction on foundation size	p. 23
K_n	the complete (simple) graph on n vertices	p. 64
$L_x(T_1)$	the leave of T_1 on point x	p. 27
M_{11}, M_{12}	the small Mathieu groups	p. 14
M_{22}, M_{23}, M_{24}	the large Mathieu groups	p. 14
m (trade)	$m(T)$, the volume of a trade	p. 2
m (general)	counter, size of collection	-
$m(T)$	volume of the trade T	p. 14
mD	a member defining set of D	p. 81
m_mD	a minimal member defining set of D	p. 81
m_sD	a smallest member defining set of D	p. 81
$MTS(v)$	Mendelsohn triple system of order v	p. 114
n (designs)	number of non-isomorphic designs, $\|\mathcal{D}\|$	p. 10
n (general)	general (counting) variable	-
N	total number of distinct designs	p. 10
N_i	total number of distinct designs in ith class	p. 10
$n(S:D)$	number of subsets of D isomorphic to S	p. 18
$P(\text{ds})$	probability that partial is a defining set	p. 141
$PG(n,q)$	projective geometry, dimension n, over field of order q	p. 13
$P(\text{min ds})$	probability that partial is a minimal defining set	p. 141
$P(\text{min}\|\text{ds})$	conditional probability partial is a minimal defining set	p. 141
$RDS(D_1, D_2)$	reduced discriminating set for D_1 and D_2	p. 101
$R_x(T_1)$	the restriction of T_1 on point x	p. 27
r	the multiplicity of a point in a design	p. 1
$r_S(T)$	multiplicity of the set S (in trade)	p. 51
$r(T)$	smallest non-zero point multiplicity in trade T	p. 51
r_x (trade)	multiplicity of the point x	pp. 27,51
$r_x(T)$ (trade)	multiplicity of the point x	p. 27
s (trade)	volume of a trade	-
s (defining set)	size of smallest defining set	-
$s(D)$	number of blocks of D with common $(k-1)$-subset	p. 74
s_i	special trade volumes between 2^t and 2^{t+1}	p. 16
(s_i, m_i)	a block and its multiplicity in a multiset	p. 96
s_i^l	generalisation of the s_i	p. 38
$\text{spec}_m(D)$	the defining spectrum	p. 21
$SQS(v)$	Steiner quadruple system on v points	p. 13
$STS(v)$	Steiner triple system on v points	p. 13
S_v	the (symmetric) group of permutation of an v-set	p. 9

Table A.3: Symbols & notation used (III)

symbol	description	reference
$\mathcal{S}[v,k,t]$	spectrum of $[v,k,t]$ trades	p. 23
$\mathcal{S}(k,t)$	spectrum of (k,t) trades	p. 23
$\mathcal{S}(t)$	spectrum of (t) trades	p. 23
$\mathcal{S}_1(k,t)$	spectrum of Steiner (k,t) trades	p. 47
$\overline{\mathcal{S}}_1(k,t)$	complement of $\mathcal{S}_1(k,t)$	p. 47
t	size of subset balanced (trade) or counted (design)	p. 1
$\bar{t}$	$\lfloor t/2 \rfloor$	p. 38
$T(M)$	$\sum_{(s_i,m_i)\in M} m_i$, total number of blocks in a multiset M	p. 97
$T_1 - T_2$	trade, difference of T_1 and T_2	p. 2
t-(v,k,λ)	parameters of a design	p. 1
v	number of points in V	p. 1
(v,k,t) trade	trade with upper bound on foundation size	p. 2
$[v,k,t]$ trade	trade with specified foundation size	p. 23
V	a generic v-set, standard set of points	p. 1
$(V,\mathcal{B})$	a generic design	p. 1
$\binom{V}{l}$	the set of all l-subsets of V	p. 73
w_d, w_t	w-sizes for minimal defining sets and trades	p. 74
xA	the blocks of A, with the point x adjoined to each	p. 26
$\lfloor x \rfloor$, $\lceil x \rceil$	floor and ceiling functions	p. 7
$\underline{x}, \underline{\underline{x}}$	new, distinct points	p. 7
$\mathbb{Z}_v$	the integers, mod v	p. 10

Table A.4: Terms, abbreviations & acronyms (I)

term	description	reference
α-resolvable	partitionable with point multiplicity α	p. 11
admissible	used of the parameters of a design	p. 9
affine geometry	residual geometry of projective geometry	p. 13
affine plane	a 2-$(q^2, q, 1)$ design	p. 13
against (DS)	a melt not in this design, but in the other	p. 99
ANSI	American National Standard Institute	-
antichain	a pairwise incomparable family	p. 74
anti-Pasch	does not contain a Pasch configuration	p. 77
automorphism	permutation preserving the set of blocks	p. 10
basic decomposition	a partition of a trade into basic subtrades	p. 41
basic (trade)	a t-trade of volume 2^t	p. 14
`bds`	standard utility for finding defining sets	p. 136
BILP	binary/Boolean/0-1 integer linear programme	p. 80
block	a k-subset of $\mathcal{B}$	p. 1
block based on G	a graph isomorphic to G	p. 171
block intersection graph	graph based on a design's block intersections	p. 64
block-residual	design produced by non-intersections	p. 12
block-transitive	all blocks in same orbit	p. 10
blockwise	containing only complete blocks	p. 22
broken pair	points paired in only half of a (partial) trade	p. 162
by complementation	a self-complementary extension	p. 12
`cad`	general-purpose completion utility, `complete`	p. 120
`cad04`	general-purpose completion utility, `comp04`	p. 120
`card`	function to find a set's cardinality	p. 121
CAT	constant amortised time	p. 78
class defining set	uniquely defines isomorphism class	p. 82
class trade	trade to a non-isomorphic design	p. 82
collection	synonym of set	p. 7
`comp03`	Steiner completion utility, $t = 2$	p. 119
`comp04`	Steiner completion utility, $t > 2$	p. 119
complementary	design formed by complementing blocks	p. 11
complete	construct design(s) containing a given partial	p. 4
`complete`	standard, general-purpose completion utility	p. 116
complete uniquely	contained in only one design	p. 4
completion	design containing given set of blocks	p. 4
configuration	a particular set of blocks	p. 71
contain	blocks of design contain (half) of a trade	p. 2
covers (RDS)	redundancy relation between RDS	p. 106
CPU	central processor unit	-
critical	ILP solution containing no smaller solution	p. 104
c-trade	class trade	p. 82
cyclic (design)	having an automorphism of order v	p. 10
decomposable	partitionable into subdesigns, same v	p. 13
decomposition (design)	a partition into configurations	p. 71
decomposition (trade)	a partition into subtrades	p. 41

TABLE A.5: Terms, abbreviations & acronyms (II)

term	description	reference
defining set	subset of design defining it uniquely	p. 3
defining spectrum	the spectrum of minimal defining sets	p. 21
degree (regular trade)	number of times each element occurs	p. 62
derangement	permutation with no fixed points	p. 85
derived	design produced by intersections	p. 12
design	a t-(v, k, λ) design	p. 1
develop	generate orbit from starter block	p. 10
difference (designs)	the difference of two designs is a trade	p. 2
difference family	a set of blocks that generates all differences	p. 11
difference set	a single block that generates all differences	p. 10
different	non-isomorphic	p. 9
discriminating set	the (multiset) difference between designs	p. 99
distinct	not equal	p. 10
DS	discriminating set	p. 99
embedded	subdesign with same λ, smaller v	p. 12
establishing set	any information that defines a unique design	p. 81
essential	if omitted, not a defining set	p. 133
equivalent (MTS)	isomorphic/same under reversal of all pairs	p. 114
excess (2-trade)	positions left after each point used twice	p. 35
extend (design)	add point to blocks, complete to $(t+1)$-design	p. 12
extension	design built by extending	p. 12
family (general)	synonym of set	p. 7
family (Steiner trades)	a collection of solely t-balanced sets	p. 49
Fano plane	the 2-$(7, 3, 1)$ projective plane of order 2	p. 13
feasible list	blocks that could be added to a partial	p. 117
feasible (swap matrix)	potentially a trade	p. 162
filter	upset	p. 76
final (design)	second member of ordered pair	p. 78
Fisher's Inequality	$b \geq v$, in a 2-design	p. 11
for (DS)	a melt in one design, but not the other	p. 99
forced (block)	there is no choice about including it	p. 3
forcing	finding forced blocks	p. 117
foundation	the set of elements in a collection of blocks	p. 14
from, to	a tradeable set of blocks in a design	p. 77
f-type (clique)	clique corresponding to a Fano plane	p. 66
full design	the design consisting of all k subsets of V	p. 11
full (orbit)	cyclic orbit of size v	p. 10
`gcc`	GNU project C compiler	p. 7
GNU	GNU's not Unix	p. 7
G-trade	a trade consisting of blocks based on G	p. 171
Hadamard design	symmetric 2-$(4m-1, 2m-1, m-1)$ design	p. 13
Hadamard 3-design	extension of Hadamard design	p. 13
half (trade)	T_1 or T_2 of a trade $T_1 - T_2$	p. 17
hole	a gap in a defining spectrum	p. 21
homogeneous	all restrictions isomorphic	p. 12
hyperplane (geometry)	n dimensional subspace	p. 13

Table A.6: Terms, abbreviations & acronyms (III)

term	description	reference
ILP	integer linear programme	p. 103
indecomposable	not decomposable	p. 13
inhomogeneous	not homogeneous	p. 12
initial (design)	first member of ordered pair, trade generation	p. 78
intersection heuristic	heuristic for building feasible list	p. 119
inversive plane	a 3-$(q^2+1, q+1, 1)$ design	p. 13
I/O	input/output	-
irreducible	indecomposable	p. 13
i-simple	any pair of blocks intersect in $\leq k-i$ points	p. 73
isomorphic	the same, up to relabelling	p. 9
k-clique	a subgraph isomorphic to K_k	p. 64
leave (trade)	the blocks not containing a point	p. 27
level (trade)	value of l in s_i^l	p. 38
line (geometry)	2 dimensional subspace	p. 13
linkage	intersection size of linked design	p. 11
linked (design)	pairwise block intersections constant size	p. 11
linked (trade)	blocks pairwise intersect in one point	p. 62
LYM Inequality	Lubell, Yamamoto & Meschalkin inequality	p. 74
mate (trade)	given T_1, any T_2 such that $T_1 - T_2$ is a trade	p. 17
Mathieu designs	design from small/large Mathieu groups	p. 14
melt	multiset element, an ordered pair (s_i, m_i)	p. 96
member trade	trade to isomorphic design	p. 81
member defining set	defines member of isomorphism class	p. 81
Mendelsohn triple system	a design with ordered pairs balanced	p. 113
minimal (defining set)	contains no smaller defining set	p. 3
minimal (trade)	contains no smaller trade	p. 2
minimal (RDS)	not covered by another RDS	p. 106
minimised (trades)	non-minimal members removed	p. 2
minimised (RDS)	collection of RDS, covered members removed	p. 106
m-trade	member trade	p. 81
multiple (design)	a design consisting of copies of another	p. 11
multiplicity	frequency of point or subset	pp. 27,51
`nauty`	the ne plus ultra of isomorphism programmes	p. 7
non-isomorphic	not isomorphic	p. 9
non-essential	can be omitted when minimising defining set	p. 133
non-simple	not simple (i.e., a (proper) multiset)	-
non-void	not empty	-
null (trade)	empty, void	p. 2
orbit	blocks/points related by an automorphism	p. 10
parallel class	set of blocks containing each point once	p. 11
parameters	t, v, k, λ, b, r (design)	p. 1
partial (design)	a family of k-subsets of V	p. 4
partially-ordered set	a set, with an order relation	p. 107

Table A.7: Terms, abbreviations & acronyms (IV)

term	description	reference
Pasch (configuration)	one half of the Pasch trade	p. 15
Pasch (trade)	the unique $(6, 3, 2)$ trade of volume 4	p. 15
PC	personal computer	p. 135
Pentium	Intel marketing-speak for the '586 processor	p. 135
plane (geometry)	3 dimensional subspace	p. 13
point (design/trade)	an element of V	p. 1
point (geometry)	1 dimensional subspace	p. 13
point-residual	design from blocks not containing a point	p. 12
point-transitive	all points in same orbit	p. 10
pointwise	containing partial blocks	p. 22
poset	partially-ordered set	p. 107
projective geometry	all subspaces of a particular vector space	p. 13
projective plane	a $2\text{-}(q^2 + q + 1, q + 1, 1)$ design	p. 13
pseudo-BILP	pseudo-binary integer linear programme	p. 103
quasi-symmetric	two possible intersection sizes	p. 11
ranked	not defined	p. 107
RDS	reduced discriminating set	p. 101
reduced discriminating set	reduced multiset difference between designs	p. 101
reducible	decomposable	p. 13
regular (trade)	all points occur the same number of times	p. 62
required list	at least one must be in any completion	p. 117
resolvable	can be partitioned into parallel classes	p. 11
restriction (design)	design from blocks containing a point	p. 12
restriction (trade)	the blocks containing a point	p. 27
rigid (design)	having a trivial automorphism group	p. 10
self-complementary	design equal to its complement	p. 11
semitriangular	starting blocks strictly increasing	p. 15
short (orbit)	cyclic orbit of size $< v$	p. 10
simple (trade/design)	having no repeated blocks (i.e., a set)	p. 2
single transposition	a permutation of the form (ij)	p. 18
single transposition free	not containing any single transpositions	p. 18
smallest (defining set)	no defining set is smaller	p. 3
smallest (trade)	a $(k + t + 1, k, t)$ trade of volume 2^t	p. 15
solely t-balanced	t-balanced, but no repeated $(t + 1)$-subset	p. 49
SPARC	scalable processor architecture	-
specifying set	defining set using block intersection sizes	p. 81
spectrum (trades)	the possible volumes	p. 23
Sperner family	antichain	p. 74
starter block	block used in developing orbit	p. 10
starting block	first block of trade, in lexicographic order	p. 15
Steiner (trade/design)	any t-subset occurs (at most) once	p. 2
Steiner quadruple system	a $3\text{-}(v, 4, 1)$ design	p. 13

Table A.8: Terms, abbreviations & acronyms (V)

term	description	reference
Steiner triple system	a 2-$(v, 3, 1)$ design	p. 13
STF	single transposition free	p. 18
s-transitive	transitive on sets of s blocks/points	p. 10
subdesign	subset of design which is also a design	p. 12
subtrade	a trade within a trade	p. 31
submelt	a melt containing fewer copies of a block	p. 97
supermelt	a melt containing more copies of a block	p. 97
support	set of distinct blocks in a collection	p. 1
support size	size of the support	p. 1
symmetric (design)	has $b = v$, $k = r$, and is linked	p. 11
swap matrix	describes distribution of elements in trade	p. 162
tail (trade)	set of points common to all blocks	p. 15
t-design	a t-(v, k, λ_t) design	p. 1
trade	a pair of disjoint and balanced collections	p. 2
trade mate	a matching half of a trade	p. 17
transitive	block-transitive	p. 10
transversal	representatives of the isomorphism classes	p. 10
trivial (design)	empty, or necessarily full/multiple	p. 11
trivial (general)	euphemism for ‘uninteresting’ or ‘boring’	-
trivial (group)	only member is the identity	p. 10
t-balanced	having the same t-subsets	p. 2
t-trade	a (v, k, t) trade	p. 14
unique (design)	there is a single isomorphism class	p. 10
unique completion	the only design containing a given set of blocks	p. 4
upset	a family that contains all supersets of its members	p. 76
wholly undetermined	a block none of whose elements are yet fixed	p. 168
viable template	a partially constructed trade	p. 162
void (trade)	empty, null	p. 2
volume (trade)	size of each collection	p. 2
v-type (clique)	clique corresponding to points	p. 66
Witt designs	design from small/large Mathieu groups	p. 14
w-size	weighted size of an antichain	p. 74

Appendix B

Summary of results

For some results on 1-trades in designs and on defining sets of 1-designs see §5.3.

Table B.1: Smallest defining set sizes, $t = 2$

design	r	b	n	$\lvert d_sD \rvert$	reference
2-(6,3,2)	5	10	1	3	[46]
2-$(6, 3, \lambda)$	$5\lambda/2$	5λ	see [27]	$3\lambda/2$	§9.1
2-(7,3,1)	3	7	1	3	[47]
2-(7,3,2)	6	14	4	6^4	[47]
2-(7,3,3)	9	21	10	7, 9^9	[47]
2-(7,3,4)	12	28	35	10, 12^{34}	§9.2
2-$(7, 3, \lambda)$	$3\lambda/7$	7λ	see [14]	$7\lambda/3$–$16\lambda/5$	§9.2
2-(8,4,3)	7	14	4	6^4	[47]
2-(9,3,1)	4	12	1	4	[46]
2-(9,3,2)	8	24	36	8^{10}, 9^{21}, 10^5	[71]
2-(9,4,3)	8	18	11	6^2, 8^9	[95]
2-(10,4,2)	6	15	3	5, 6, 8	[52]
2-(10,4,4)	12	30	>998	16 (3)	[52]
2-(10,5,4)	9	18	21	6^2, 7^{18}, 8	[74, 100], §7.1.5
2-(11,5,2)	5	11	1	5	[46]
2-(13,3,1)	6	26	2	8, 9	[51]
2-(13,4,1)	4	13	1	6	[46]
2-(15,3,1)	7	35	80	11^{52}, 12^{20}, 13^4, 14^3, 16	[93, 99], §11.1
2-(15,7,3)	7	15	5	7^3, 8, 9	[48, 49]
2-(16,4,1)	5	20	1	7	[46]
2-(16,6,2)	6	16	3	7^2, 9	[52]
2-(19,9,4)	9	19	6	8^6	[94]
2-(21,5,1)	5	21	1	8	[46]
2-(25,5,1)	6	30	1	10	[44]
2-(23,11,5)	11	23	1103	8 (q)	[94]
2-(27,13,6)	13	27	≥ 7	≤ 11 (q)	[94]
2-(31,6,1)	6	31	1	11	[44]
2-(31,15,7)	15	31	$\geq 10^6$	≤ 10 (q)	[94]
				24 (pg)	[38]
2-(63,31,15)	31	63	$\geq 10^{17}$	52–55 (pg)	[38]

(3): the 3-$(10, 4, 1)$ as a 2-design
(q): the Hadamard design from the quadratic residues
(pg): the design associated with PG(d,2)

Table B.2: Smallest defining set sizes, $t = 3$

design	r	b	n	$\|d_sD\|$	reference
3-(8,4,1)	7	14	1	3	[47]
3-(8,4,2)	14	28	4	6^4	[47]
3-(8,4,3)	21	42	10	$7, 9^9$	[47]
3-(8,4,4)	28	56	31	$10, 12^{30}$	§9.3
3-$(8, 4, \lambda)$	7λ	14λ	see §9.3	$7\lambda/3$–$16\lambda/5$	§9.3
3-(10,4,1)	12	30	1	4	[52]
3-(10,5,3)	18	36	7	$5, 6^2, 8^4$	[95, 100], §7.1.4
3-(12,6,2)	11	22	1	5	[94]
3-(16,8,3)	15	30	5	$7^3, 8, 9$	[94]
3-(17,5,1)	20	68	1	7	§11.3
3-(20,10,4)	19	38	3	8^3	[94]
3-(22,6,1)	21	77	1	8	[93]
3-(24,12,5)	23	46	≥ 1	8 (q)	[94]
3-(28,14,6)	27	54	≥ 1	≤ 11 (q)	[94]
3-(32,16,7)	31	62	≥ 1	≤ 10 (q)	[94]

(q): the Hadamard design from the quadratic residues

Table B.3: Smallest defining set sizes, $t \geq 4$

design	r	b	n	$\|d_sD\|$	reference
4-(11,5,1)	30	66	1	5	[52]
4-(23,7,1)	77	253	1	8	[93]
5-(12,6,1)	66	132	1	5	[52]
5-(24,8,1)	253	759	1	8	[16, 93]

Table B.4: Other results on defining set sizes

design	$\|m_sD\|$	$\|c_sD\|$	establishing	reference
2-$(6, 3, \lambda)$	√	√	-	§9.1
2-(7,3,3)	-	-	within transversal	§8.4
2-$(7, 3, \lambda)$	√	√	-	§9.2
2-(8,4,3)	√	√	-	§7.1.1
2-(9,3,2)	-	-	simple/non-simple	§9.4
2-(9,4,3)	√	√	-	§7.1.3
2-(10,4,2)	√	√	-	§7.1.2
2-(10,5,4)	√	√	-	§7.1.5
3-$(8, 4, \lambda)$	√	√	-	§9.3
3-(10,5,3)	√	√	-	§7.1.4

Table B.5: Distribution of smallest/minimal defining sets

design	description	reference
2-(7,3,1)	minimal defining sets (all)	§11.4
2-(9,3,1)	minimal defining sets (all)	§11.4
2-(13,3,1)	minimal defining sets (all)	§11.4
2-(13,4,1)	minimal defining sets (all)	§11.4
2-(15,3,1)	minimal defining sets (sampled)	§11.2
2-(16,4,1)	minimal defining sets (all)	§11.4
2-(21,5,1)	minimal defining sets (all)	§11.4
2-(25,5,1)	minimal defining sets (sampled)	§11.4
2-(31,6,1)	minimal defining sets (sampled)	§11.4
3-(8,4,1)	minimal defining sets (all)	§11.4
3-(10,4,1)	minimal defining sets (all)	§11.3
3-(17,5,1)	smallest defining sets (all)	§11.3
	minimal defining sets (sampled)	§11.3
4-(11,5,1)	smallest defining sets (all)	§11.3
	minimal defining sets (partial count)	§11.3
5-(12,6,1)	smallest defining sets (all)	§11.3

Table B.6: Trades in designs

design	description	reference
2-(7,3,1)	all trades by volume/foundation	§7.4
2-(8,4,3)	all trades by volume/foundation	§7.4
2-(9,3,1)	all trades by volume/foundation	§7.4
2-(13,3,1)	Pasch trades	§11.4
2-(10,4,2)	all trades by volume/foundation	§7.4
2-(15,3,1)	number of Pasch trades	§11.2
2-(15,3,1)	disjoint Pasch trades	§10.6.1
2-(19,3,1)	disjoint Pasch trades	§10.6.1
2-(21,5,1)	discussion	§11.4
2-(25,5,1)	discussion	§11.4
3-(8,4,1)	all trades by volume/foundation	§7.4
3-(10,4,1)	all trades by volume/foundation	§7.4
3-(17,5,1)	known trade volumes	§7.5
4-(11,5,1)	all trades by volume/foundation	§7.4

▷ See also Appendix E, and Chapter 7

Appendix C

The swap matrix technique (Chapter 4)

The proofs in this appendix use what we term the *swap matrix* technique. Suppose that an element x has multiplicity r_x in a Steiner $(k, 2)$ trade $T = T_1 - T_2$. The idea is to list the possible arrangements of elements in the blocks containing x in T_1 and T_2.

Recall that the $r_x(k-1)$ elements with which x occurs must all be distinct. Since $T_1 \cap T_2 = \emptyset$, the $2r_x$ blocks in $T_1 \cup T_2$ containing x must all be distinct. Label the blocks of each of T_1 and T_2 containing x with $1, 2, \ldots, r_x$. Let

$$S = [s_{ij}],\ 1 \leq i, j \leq r_x,$$

be an $r_x \times r_x$ matrix where s_{ij} is the number of elements, other than x, from block j of T_1 that appear in block i of T_2. S is called a **swap matrix** and it is simple to see that $0 \leq s_{ij} \leq k-2$ and the row and column sums of S equal $k-1$. Thus, each row and column of S is a partition of $k-1$ into r_x parts, at least two of which are non-zero.

Any swap matrix can occur in many distinct but equivalent forms. Permutations of rows and columns lead to equivalent swap matrices. Permuting rows is equivalent to reordering the blocks of T_2 and permuting columns to reordering the blocks of T_1. Transposing S is equivalent to exchanging T_1 and T_2. We will always use a form where $s_{11} \geq s_{ij}$ and $s_{12} \geq s_{21}$, and where the first row and the first column are non-increasing. Note that this is not sufficient to eliminate all equivalences but suffices in the cases we consider.

Suppose that $xab \subseteq B \in T_1$, and that no block of T_2 which contains x also contains both a and b. Then the pair ab is called a **broken pair**. A swap matrix is said to be **feasible** if all the broken pairs can be contained, as partial blocks, in the $m(T) - r_x$ remaining blocks of T_2 without violating the Steiner property. Given a feasible swap matrix, an apportioning of the broken pairs to the remaining blocks of T_2 is called a **viable template**. Note that a feasible swap matrix may yield more than one viable template, and that a viable template is not necessarily extendable to a Steiner trade.

We will determine possible swap matrices for Steiner $(k, 2)$ trades of volume m for specific values of k and m. In the first two sections of this appendix, these are used to prove non-existence for the cases where $k = 5$, $m = 11$ and $k = 6$, $m = 13$. In the final section, they are used to show structural uniqueness for the case where $k = 5$, $m = 10$.

C.1 Steiner $(5, 2)$ trades of volume 11

For Steiner (5,2) trades, the only volume where existence is not yet settled is eleven. We will assume that such a trade, $T = T_1 - T_2$, exists and obtain a contradiction. Including repetitions, there are $11 \times 5 = 55$ elements in T_1 and hence there must be an element in $F(T)$ with odd multiplicity. Since each element must occur at least twice and no more than six times, by Lemma 4.14, there must be an element with multiplicity three or five. We will show that both cases are impossible.

Lemma C.1: *If T is a Steiner $(5, 2)$ trade with $m(T) < 12$, then there is no element with multiplicity three.*

Proof: Let $x \in F(T)$ and $r_x = 3$. The possible partitions in a swap matrix for x are {3,1,0}, {2,2,0} and {2,1,1}. It is easy to see that, up to equivalence, there are only six possible swap matrices. These are shown in the first column of Table C.1. Let the elements occurring with x be 1, 2, …, 9, a, b, c. The column headed T_2 is obtained by applying the swap matrix to T_1. The broken pairs are listed in the final column. Each row of broken pairs comes from the corresponding block of T_1. We will show that none of these swap matrices is feasible in less than 12 blocks.

For a given swap matrix, any two broken pairs from two different rows contain four distinct elements. Since there are only three rows containing x, two such broken pairs cannot appear together in a block in T_2 without duplicating a pair already in the blocks of T_2.

The rows of broken pairs labelled with a star are generated by columns of the swap matrix with a partition of {3,1,0} or {2,2,0}. These rows have the property that no two broken pairs in the same row can occur in the same block of T_2 without violating the Steiner propery. Thus we see that the third and fourth swap matrices imply that there are at least twelve and fifteen blocks in T_2 respectively. Similarly,

Table C.1: The swap matrices for Steiner (5,2) trades with $r_x = 3$

Swap Matrix	No.	T_1	T_2	Broken Pairs
3 1 0	1	x 1 2 3 4	x 1 2 3 5	$14, 24, 34^*$
1 2 1		x 5 6 7 8	x 4 6 7 9	$56, 57, 58, 68, 78$
0 1 3		x 9 a b c	x 8 a b c	$9a, 9b, 9c^*$
3 1 0	2	x 1 2 3 4	x 1 2 3 5	$14, 24, 34^*$
1 1 2		x 5 6 7 8	x 4 6 9 a	$56, 57, 58, 67, 68$
0 2 2		x 9 a b c	x 7 8 b c	$9b, 9c, ab, ac^*$
3 1 0	3	x 1 2 3 4	x 1 2 3 5	$14, 24, 34^*$
1 0 3		x 5 6 7 8	x 4 9 c b	$56, 57, 58^*$
0 3 1		x 9 a b c	x 6 7 8 c	$9c, ac, bc^*$
2 2 0	4	x 1 2 3 4	x 1 2 5 6	$13, 14, 23, 24^*$
2 0 2		x 5 6 7 8	x 3 4 9 a	$57, 58, 67, 68^*$
0 2 2		x 9 a b c	x 7 8 b c	$9b, 9c, ab, ac^*$
2 2 0	5	x 1 2 3 4	x 1 2 5 6	$13, 14, 23, 24, 34$
1 1 2		x 5 6 7 8	x 3 7 9 a	$57, 58, 67, 68, 78$
1 1 2		x 9 a b c	x 4 8 b c	$9b, 9c, ab, ac^*$
2 1 1	6	x 1 2 3 4	x 1 2 5 9	$13, 14, 23, 24, 34$
1 2 1		x 5 6 7 8	x 3 6 7 a	$56, 57, 58, 68, 78$
1 1 2		x 9 a b c	x 4 8 b c	$9a, 9b, 9c, ab, ac$

to generate a trade in less than twelve sets from the second swap matrix would require that all of the pairs in the second row occurred in the same set, which would repeat the pair 78 in T_2.

The remaining three swap matrices all contain rows of five broken pairs derived from the partition {2,1,1}. All these sets of five broken pairs are isomorphic to the set {56, 57, 58, 68, 78} from swap matrix No. 1. The pair 67 already appears in T_2, so the pairs 56 and 57 must occur in different blocks in T_2, as must the pairs 68 and 78. If we used the two partial blocks 568 and 578 to cover all broken pairs, the pair 58 would occur twice. Thus each row of broken pairs without a star requires at least three blocks in T_2.

The first, fifth and sixth swap matrices would require an additional nine, ten and nine blocks respectively in T_2. Thus the total number of blocks in T_2 is greater than eleven. □

Lemma C.2: *If T is a Steiner $(5, 2)$ trade of volume eleven, then there is no element with multiplicity six.*

Proof: Suppose $y \in F(T)$ and $r_y = 6$. Then y is paired with $6(k-1) = 24$ distinct elements. Each of these elements must occur at least once in the five blocks not containing y, leaving exactly one position of these blocks unaccounted for. This position cannot contain a new element, since it would have multiplicity one. Thus one of the elements occurring with y must have multiplicity three, contradicting Lemma C.1. $\square$

As the number of elements with odd multiplicity is necessarily odd and because there is no element with multiplicity three, there must be an odd number of elements with multiplicity five. It is easy to see that three or more elements with multiplicity five cannot be contained in eleven blocks without duplicating a pair. Thus there must be exactly one element with multiplicity five.

For $2 \leq i \leq 6$, let n_i denote the number of elements of $F(T)$ that have multiplicity i. We have shown that $n_3 = n_6 = 0$ and that $n_5 = 1$. Thus $2n_2 + 4n_4 + 5 = 55$; that is, $n_2 + 2n_4 = 25$. Given that $n_5 = 1$, it is straightforward to check that there are at most four elements with multiplicity four; that is, $n_4 \leq 4$.

The following analogue of Lemma 4.16 is needed to complete our analysis.

LEMMA C.3: *Let $T = T_1 - T_2$ be a Steiner $(5,2)$ trade with $r_\alpha = 2$ for some $\alpha \in F(T)$, and suppose that $\sum_{i \geq 3} n_i \leq 7$. If B_1 and B_2 are the two blocks of T_1 containing α, then there exist (distinct) elements $x \in B_1$ and $y \in B_2$ such that at least four blocks of T_1 contain x but not y and at least four blocks contain y but not x.*

PROOF: The possible partitions are {3,1} and {2,2} and there are only two inequivalent swap matrices for α,

$$S_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Without loss of generality, let $B_1 = \alpha 1234$ and $B_2 = \alpha 5678$. We first prove that S_2 is not feasible.

If S_2 is feasible then, without loss of generality, $\alpha 1256$ and $\alpha 3478$ are in T_2. This yields the set of broken pairs {13, 14, 23, 24, 57, 58, 67, 68}. Now, no two of these pairs can occur together in T_2, since the trade is Steiner. Thus each of the elements $1, \ldots, 8$ must occur at least three times in T_2, contradicting $\sum_{i \geq 3} n_i \leq 7$.

We now show S_1 is feasible and yields a single viable template. We can assume, without loss of generality, that $\alpha 1235$ and $\alpha 4678$ are in T_2. This yields the set of broken pairs {14, 24, 34, 56, 57, 58}. Now, no two of these pairs can occur together in T_2, since the trade is Steiner. Thus both $x = 4$ and $y = 5$ occur in at least four blocks that do not contain the other element. $\square$

THEOREM C.4: *If T is a Steiner $(5, 2)$ trade, then $m(T) \neq 11$.*

PROOF: Assume that $T = T_1 - T_2$ is a Steiner $(5, 2)$ trade of volume eleven. We will derive a contradiction.

The only integral solutions (n_2, n_4) to $n_2 + 2n_4 = 25$ satisfying the necessary condition $n_4 \leq 4$ are (17,4), (19,3), (21,2), (23,1) and (25,0). The case (25,0) is eliminated immediately by Lemma C.3. Recall that $n_5 = 1$ and note that $n_4 > 0$ in all remaining cases. Let x be the element with multiplicity five. Since $n_4 \leq 4$, there must be at least one block, say B, which contains x and none of the multiplicity four elements.

First, suppose that each of the elements of multiplicity four occurs in some (but not necessarily the same) block in which x occurs. The four elements that occur with x in B must be multiplicity two elements. Thus the elements of multiplicity at least four that these elements are paired with must be x and one of the multiplicity four elements. But all multiplicity four elements are paired with x, contradicting Lemma C.3.

Thus there exists an element, say y, with multiplicity four which does not occur in a block with x. Let U and V be the two blocks containing neither x nor y. Consider the $5(k - 1) = 20$ distinct elements contained in a block with x. Since $n_4 \leq 4$, at least 17 of these have multiplicity two. Now at most $4(k - 1) = 16$ of these can occur in blocks containing y. Thus one of them, say z, occurs in one of U or V, say U. By Lemma C.3, z occurs with two elements, say α and β, each of which occurs in four blocks not containing the other. Now, at least one of α or β, say α, must have multiplicity four and be contained in U. The other three occurrences of α must be in V, in a block with y, and a block with x (not that containing z). Now consider β. We cannot have $\beta = x$, since we already have the pair αx, contradicting Lemma C.3. Thus $r_\beta = 4$, and it must occur in the block containing xz, in a block with y, and in both U and V. But the pair $\alpha\beta$ is repeated and this completes the proof. □

C.2 Steiner $(6, 2)$ trades of volume 13

The only volume for Steiner (6,2) trades whose existence is unresolved is thirteen. Such a trade would have a total of 78 elements in T_1, with possible element multiplicities of $r = 2, \ldots, 7$. Exhaustive searches by computer programmes based on

TABLE C.2: The swap matrices and templates for Steiner (6,2) trades

r	Swap Matrix	T_1	T_2
2	4 1	x 1 2 3 4 5	x 1 2 3 4 6
	1 4	x 6 7 8 9 a	x 5 7 8 9 a
			1 5 * * * *
			2 5 * * * *
			3 5 * * * *
			4 5 * * * *
			6 7 * * * *
			6 8 * * * *
			6 9 * * * *
			6 a * * * *
			* * * * * *
			* * * * * *
			* * * * * *
5	1 1 1 1 1	x 1 2 3 4 5	x 1 6 b g l
	1 1 1 1 1	x 6 7 8 9 a	x 2 7 c h m
	1 1 1 1 1	x b c d e f	x 3 8 d i n
	1 1 1 1 1	x g h i j k	x 4 9 e j o
	1 1 1 1 1	x l m n o p	x 5 a f k p
			1 2 3 4 5 *
			6 7 8 9 a *
			b c d e f *
			g h i j k *
			l m n o p *
			* * * * * *
			* * * * * *
			* * * * * *
6	1 1 1 1 1 0	x 1 2 3 4 5	x 1 6 b g l
	1 1 1 1 0 1	x 6 7 8 9 a	x 2 7 c h q
	1 1 1 0 1 1	x b c d e f	x 3 8 d m r
	1 1 0 1 1 1	x g h i j k	x 4 9 i n s
	1 0 1 1 1 1	x l m n o p	x 5 e j o t
	0 1 1 1 1 1	x q r s t u	x a f k p u
			1 2 3 4 5 *
			6 7 8 9 a *
			b c d e f *
			g h i j k *
			l m n o p *
			q r s t u *
			* * * * * *

the swap matrix method yielded a single equivalence class of feasible swap matrices for each of $r = 2$, 5 and 6, and showed that none existed for $r = 3$ or 4. Each feasible swap matrix yielded a single viable template. The matrices, with T_1 and the templates for T_2, are shown in Table C.2. Note that all three of the swap matrices are symmetric, so the templates for T_1 are isomorphic to those for T_2. Using these templates, we will show that it is not possible to construct a Steiner $(6, 2)$ trades of volume thirteen.

Lemma C.5: *If $T = T_1 - T_2$ is a Steiner $(6, 2)$ trade of volume thirteen, then there is no element with multiplicity seven.*

Proof: Suppose there exists an element of multiplicity seven in T_1. Such an element is paired with $7(k-1) = 35$ distinct other elements. Each of these 35 elements must occur at least once more which leaves one position of the trade unaccounted for. This remaining position must be filled by an element of multiplicity one or three. However, multiplicity one is not possible and there are no viable templates for $r = 3$. This completes the proof. □

Theorem C.6: *If T is a Steiner $(6, 2)$ trade, then $m(T) \neq 13$.*

Proof: Let n_2, n_5 and n_6 denote the number of elements of $F(T)$ that have multiplicity 2, 5 and 6 respectively. Then $2n_2 + 5n_5 + 6n_6 = 78$. Now it is easy to see that $n_5 \leq 3$ and $n_6 \leq 2$, else the Steiner property is violated. Further, the feasible template for $r = 2$ shows that, if $n_2 \geq 1$, then $n_5 + n_6 \geq 2$. Considering all these equations and inequalities, and noting that n_5 must be even, we see that the only integral solutions (n_2, n_5, n_6) are (28,2,2), (31,2,1), (33,0,2) and (34,2,0). Note that $f(T) = n_2 + n_5 + n_6$, and that the templates use 11, 26 and 31 distinct elements respectively. The last three, three and one blocks of these templates respectively are said to be **wholly undetermined**.

Case (28,2,2): Clearly this contradicts the Steiner property.

Case (31,2,1): Three distinct new elements must be added to the multiplicity six template to reach the final foundation size of 34. At most two of the existing elements could occur again (as multiplicity five elements). Thus there are at most five distinct elements unplaced to fill the six places of the wholly undetermined block in the template.

Case (33,0,2): Four distinct new elements must be added to the multiplicity six template to reach the final foundation size of 35. At most one of the existing elements could occur again (as a multiplicity six element). Thus there are at most five distinct elements unplaced to fill the six places of the wholly undetermined block in the template.

Case (34,2,0): The template for multiplicity two shows that the two elements that occur five times each do not occur together. Thus, in the template for multiplicity five the element other than x of multiplicity five, say y, is distinct from $1, \ldots, p$. Now at least two of the five occurrences of y must be in the partially filled sets of T_2.

By the symmetry of the swap matrix, it is easy to see that the second occurrences of the five elements, distinct from y, of such a set are in separate sets in T_1. Thus, to balance pairs, all five occurrences of y in T_1 and in T_2 must be with the partial sets in the templates for T_1 and T_2. But now the ten sets containing x or y in T_1 and in T_2 form a Steiner $(6,2)$ subtrade of volume ten. Thus the three remaining sets in T_1 and in T_2 must form a Steiner $(6,2)$ subtrade of volume $3 < 2(k-1) = 10$, which is impossible by Theorem 4.18. □

C.3 Steiner $(5,2)$ trades of volume 10

Suppose that $T = T_1 - T_2$ is a Steiner $(5,2)$ trade of volume ten. We will prove that T has the structure given in Lemma 4.20(3). By Lemma 4.14, $2 \leq r_x \leq 5$ for all $x \in F(T)$. For $2 \leq i \leq 5$, let n_i denote the number of elements in $F(T)$ that have multiplicity i. Then $2n_2 + 3n_3 + 4n_4 + 5n_5 = 50$. By Lemma C.1, $n_3 = 0$, and it is easy to see that $n_5 \geq 3$ is not possible, since T is Steiner. So, since n_5 must be even, we need only consider the cases $n_5 = 0$ or 2.

Lemma C.7: *If $n_5 = 2$, then $T_1 = xA^1 + yA^2$ and $T_2 = xA^2 + yA^1$, where A^1 and A^2 are solely 1-balanced and $x, y \notin F(A^1)$.*

Proof: This follows immediately from Lemma 4.17. □

Lemma C.8: *If $n_5 = 0$, then T does not exist.*

Proof: Since $n_3 = n_5 = 0$, then $n_2 + 2n_4 = 25$. It is straightforward to check that $n_4 \leq 5$, and that $n_4 = 4$ or 5 requires that any two multiplicity four elements occur together in a block. Since $n_2 > 0$ in all the remaining cases, Lemma C.3 implies that $n_4 \geq 2$ and that the $n_4 = 4$ or 5 cases are not possible. So $n_4 = 2$ or 3. We now use the unique multiplicity two template for T_2 given in the proof of Lemma C.3, with foundation size nine and elements 4 and 5 of multiplicity four. Since $m(T) = 10$, there are two blocks, say U and V, which are wholly undetermined.

Case $n_2 = 21$, $n_4 = 2$: Here $f(T) = 23$, so fourteen new elements are to be added to the multiplicity two template. By Lemma C.3, each of these multiplicity two elements has to occur with both of the multiplicity four elements, which is impossible.

Case $n_2 = 19$, $n_4 = 3$: Here $f(T) = 22$, so thirteen new elements are required. If each of these has multiplicity two, then the argument of the previous case applies. So one of these elements, say α, has multiplicity four. Now α must occur in both

U and V, in a block with 4, and a block with 5. Suppose β is a new multiplicity two element that occurs in the block U. One of the multiplicity four elements that β occurs with must be α. But α occurs with both of the other multiplicity four elements, contradicting Lemma C.3. $\square$

Appendix D

G-trades (Chapter 4)

A block in a $(k, 2)$ trade can be viewed as a complete graph of order k, K_k, with edges representing pairs. In [8] the trade spectrum problem is generalised to trades based on arbitary simple graphs. In this appendix we briefly indicate how the techniques developed in Chapter 4 can be applied to this problem, and prove a strengthened version of Theorem 3.2 of [8]. Let G be a simple (cf. Steiner property) graph with $\nu(G)$ labelled vertices. We call a graph isomorphic to G a **block based on G**.

Definition D.1: *Let $T_1 = \{G_1, G_2, \ldots, G_m\}$ and $T_2 = \{G'_1, G'_2, \ldots, G'_m\}$ each be collections of m blocks based on G. If $\bigcup_{i=1}^{m} G_i = \bigcup_{i=1}^{m} G'_i$ is a simple graph and $G_i \neq G'_j$, $1 \leq i, j \leq m$, then $T = T_1 - T_2$ is a **G-trade** of volume m.*

Example D.2: *In Figure D.1, G is a path of length three and T_1 and T_2 are disjoint decompositions of K_4 into blocks based on G.* □

The construction method of Lemma 4.3(1) fails for general G-trades. For instance, in Example D.2, relabelling all the elements of any block of T_1 relabels all the elements of the trade. However, in what follows, we construct blocks based on G in a particular manner, which allows us to apply Lemma 4.3(1).

For the remainder of this appendix the minimum degree of G, $\delta(G)$, is at least two. Let $\delta(G) = k - 1$. For this value of k the collections R, C, F and $\underline{R}_r, \underline{C}_r$ are defined as in Section 4.2, with the understanding that the rows (that is, $(k-1)$-subsets) will be used to label vertices of blocks based on G.

Figure D.1: An example G-trade

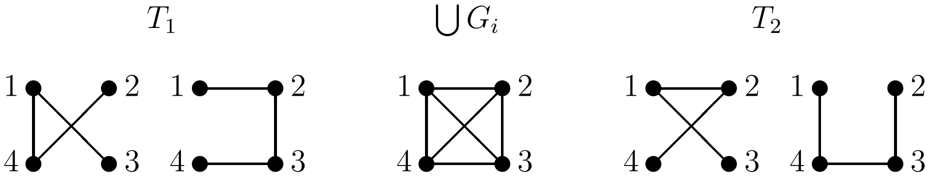
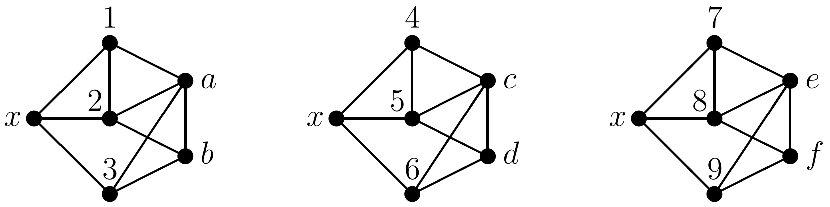


FIGURE D.2: The three blocks of (x, R, D)



We use triples of the form (x, N, D) to represent $k-1$ blocks based on G constructed as follows. Here, x represents a fixed vertex of minimum degree $k-1$, $N \in \{R, C, F, \underline{R}_r, \underline{C}_r\}$ and D is a collection of $k-1$ disjoint sets (rows) each of $\nu(G)-k$ vertices. The $k-1$ blocks based on G are formed by placing a copy of G on x and the ordered ith rows of N and D in a consistent manner; ensuring, for instance, that the neighbourhood of x equals the ith row of N for each $i = 1, \ldots, k-1$, and that two vertices from the same positions of N and D in different blocks are either always adjacent or non-adjacent depending on how the first copy of G was placed.

EXAMPLE D.3: *In Figure D.2, our notation is illustrated for a graph G with $\delta(G) = 3$. So $k = 4$ and $R = \{123, 456, 789\}$. If we choose $D = \{ab, cd, ef\}$ as shown, then the three blocks based on G can be written as (x, R, D).* □

We will now show how our constructions in Section 4.3 can be applied to G-trades. The following result is analogous to Lemma 4.24. Recall that $\underline{A}_r$ represents A with the elements a_{ij} of the first r rows of A relabelled to $\underline{a}_{ij}$. $\underline{D}_r$ is defined similarly.

LEMMA D.4: *There exists a G-trade of volume $3(k-1)+r$ for all $0 \leq r \leq k-1$.*

PROOF: Let

$$\begin{aligned} T^a &= T_1^a - T_2^a = +(x, R, D^1) + (y, F, D^2) - (x, F, D^2) - (y, R, D^1), \\ T^b &= T_1^b - T_2^b = +(x, \underline{C}_r, D^3) + (z, \underline{R}_r, \underline{D}_r^1) - (z, \underline{C}_r, D^3) - (x, \underline{R}_r, \underline{D}_r^1), \end{aligned}$$

where the $F(D^i)$ are mutually disjoint, $F(D^i \cup \underline{D}_r^i) \cap F(R \cup \underline{R}_r) = \emptyset$ for $i = 1, 2, 3$ and $x, y, z \notin F(D^1 \cup \underline{D}_r^1 \cup D^2 \cup D^3 \cup R \cup \underline{R}_r)$.

Then $T^a + T^b$ is a G-trade if each edge in the positive half of $T^a + T^b$ occurs precisely once. Suppose that there exists an edge e in the positive half of $T^a + T^b$ occurring twice. It is immediate that e is not incident with y, z or any vertices of D^2 or D^3. Additionally e is not in F. Therefore, e is necessarily in (x, R, D^1) and $(x, \underline{C}_r, D^3) \cup (z, \underline{R}_r, \underline{D}_r^1)$. However, this implies that e is in one of the blocks of (x, R, D^1) that cancels. Hence $T^a + T^b$ is a G-trade of the required volume. □

A lemma for G-trades analogous to Lemmas 4.23 is similarly proved, while the analogue of Lemma 4.3(2) is trivial. Although, as we have noted, there is no general analogue of Lemma 4.3(1), our construction of blocks based on G, utilising the solely 1-balanced families and mutually disjoint D^i, allows us to apply a restricted version of this result. So we can also prove an analogue of Lemma 4.25. Combining these results we obtain the following.

THEOREM D.5: *Let G be a simple graph with minimum degree $\delta(G) = \delta \geq 2$. If $m \geq 3\delta$, or $m \geq 2\delta$ and m is even, then there exists a G-trade of volume m.* □

This is a significant improvement on Theorem 3.2 of [8], which proves the existence of G-trades for volumes of at least $5\delta + \varepsilon$, where $\varepsilon \in \{-3, -1, 0, +2\}$ depends on the congruence class of $\delta(G)$ modulo 6.

APPENDIX E

Tables for Chapter 7

TABLE E.1: The number of distinct trades in the 2-$(8,4,3)$ designs

initial	final design					
design	α^*	β^*	γ^*	δ^*	c-trades	total
α^*	981	343	27	819	983	1508
β^*	710	251	22	535	1088	1203
γ^*	428	154	15	478	1046	1053
δ^*	1172	379	30	848	1214	1599

TABLE E.2: The number of minimal trades in the 2-$(8,4,3)$ designs

initial	final design					
design	α^*	β^*	γ^*	δ^*	c-trades	total
α^*	136	6	5	68	62	28
β^*	120	72	2	112	122	32
γ^*	112	56	7	16	56	56
δ^*	14	35	16	133	14	70

TABLE E.3: Smallest defining set sizes of the 2-$(8,4,3)$ designs

design	A_i	N_i	$\log_5 N_i$	$\lvert m_sD\rvert$	$\lvert c_sD\rvert$	$\lvert d_sD\rvert$
α^*	12	3360	5.05	5	6	6
β^*	48	840	4.18	4	6	6
γ^*	1344	30	2.11	3	6	6
δ^*	21	1920	4.70	5	6	6

Table E.4: The number of distinct trades in the 2-$(10, 4, 2)$ designs

initial design	final design H_1	H_2	H_3	c-trades	total
H_1	66	546	1401	1461	1461
H_2	133	900	1879	1882	2049
H_3	228	1268	2102	1279	2363

Table E.5: The number of minimal trades in the 2-$(10, 4, 2)$ designs

initial design	final design H_1	H_2	H_3	c-trades	total
H_1	15	45	255	45	45
H_2	11	135	18	21	21
H_3	23	9	221	9	105

Table E.6: Smallest defining set sizes of the 2-$(10, 4, 2)$ designs

design	A_i	N_i	$\log_{14} N_i$	$\lvert m_sD\rvert$	$\lvert c_sD\rvert$	$\lvert d_sD\rvert$
H_1	720	5040	3.23	4	8	8
H_2	48	75600	4.26	5	6	6
H_3	24	151200	4.51	5	5	5

Table E.7: The number of distinct trades in the 2-$(9, 4, 3)$ designs

initial	final design												
design	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$	$\mathcal{M}_9$	$\mathcal{M}_{10}$	$\mathcal{M}_{11}$	c-trades	total
$\mathcal{M}_1$	994	4423	12616	7981	17638	17134	9256	6412	3016	21076	5737	35293	35293
$\mathcal{M}_2$	1071	4506	14266	6641	21996	15262	8894	6704	3248	20722	7017	35277	35293
$\mathcal{M}_3$	1107	4967	15583	6869	21646	15163	8968	7089	3335	21563	7175	34164	35821
$\mathcal{M}_4$	1082	4912	14647	7524	19791	15894	9300	6852	3278	21686	6466	35267	35821
$\mathcal{M}_5$	1119	5065	15803	6975	21894	15354	9054	7214	3333	21392	7251	32087	35889
$\mathcal{M}_6$	1131	5053	15302	7322	21066	15774	9231	7061	3325	21481	6866	34026	35889
$\mathcal{M}_7$	1146	5026	15306	7309	20937	15693	9070	7084	3336	21447	6906	35319	35889
$\mathcal{M}_8$	980	4492	15274	6816	21449	14972	8821	7152	3234	20524	7141	33324	34228
$\mathcal{M}_9$	1000	4492	14204	7652	19532	16596	8956	7048	3268	20940	6455	33908	34228
$\mathcal{M}_{10}$	1143	5122	15857	7247	21292	15414	9099	7156	3326	21663	7083	32388	36002
$\mathcal{M}_{11}$	1081	5092	16144	6766	22036	14935	8941	7351	3361	21490	7409	35596	36002

Table E.8: The number of minimal trades in the 2-$(9, 4, 3)$ designs

initial	final design												
design	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$	$\mathcal{M}_9$	$\mathcal{M}_{10}$	$\mathcal{M}_{11}$	c-trades	total
$\mathcal{M}_1$	342	342	2592	54	3168	1080	1032	828	486	2844	828	378	378
$\mathcal{M}_2$	214	536	64	510	1136	1248	944	644	558	2676	896	378	378
$\mathcal{M}_3$	213	551	1204	770	330	933	838	627	522	1114	587	298	298
$\mathcal{M}_4$	107	663	1340	620	2046	300	604	720	390	1140	894	298	298
$\mathcal{M}_5$	218	699	742	832	1142	894	765	648	528	441	457	282	294
$\mathcal{M}_6$	148	748	1100	371	1117	885	714	696	474	343	708	286	294
$\mathcal{M}_7$	164	717	972	459	1047	885	958	687	477	339	625	294	294
$\mathcal{M}_8$	228	616	1400	824	1596	1418	1122	295	402	1592	972	978	757
$\mathcal{M}_9$	192	512	1280	624	1648	1456	1024	312	718	1632	752	808	757
$\mathcal{M}_{10}$	183	762	1025	627	417	511	646	681	506	993	564	288	298
$\mathcal{M}_{11}$	267	756	990	1134	189	1476	945	639	540	936	871	288	298

Table E.9: Smallest defining set sizes of the 2-$(9, 4, 3)$ designs

design	A_i	N_i	$\log_7 N_i$	$\lvert m_s D\rvert$	$\lvert c_s D\rvert$	$\lvert d_s D\rvert$
$\mathcal{M}_1$	144	2520	4.03	4	8	8
$\mathcal{M}_2$	16	22680	5.15	5	8	8
$\mathcal{M}_3$	2	181440	6.22	5	8	8
$\mathcal{M}_4$	8	45360	5.51	5	8	8
$\mathcal{M}_5$	1	362880	6.58	6	8	8
$\mathcal{M}_6$	2	181440	6.22	5	8	8
$\mathcal{M}_7$	6	60480	5.66	5	8	8
$\mathcal{M}_8$	8	45360	5.51	5	4	6
$\mathcal{M}_9$	32	11340	4.80	5	4	6
$\mathcal{M}_{10}$	1	362880	6.58	6	8	8
$\mathcal{M}_{11}$	9	40320	5.45	5	8	8

Table E.10: The number of distinct trades in the 3-$(10, 5, 3)$ designs

initial	final design								
design	$\mathcal{N}_1$	$\mathcal{N}_2$	$\mathcal{N}_3$	$\mathcal{N}_4$	$\mathcal{N}_5$	$\mathcal{N}_6$	$\mathcal{N}_7$	c-trades	total
$\mathcal{N}_1$	3765	20151	130561	280051	9487	38296	200371	446232	448287
$\mathcal{N}_2$	4311	4723	15541	27418	7138	7456	21634	46245	46263
$\mathcal{N}_3$	4455	5291	17243	26764	8170	7792	22548	46113	48171
$\mathcal{N}_4$	4549	5393	17514	26991	8581	7915	22421	42006	48735
$\mathcal{N}_5$	3755	8856	62192	107907	7928	20752	81667	149986	156932
$\mathcal{N}_6$	3763	4724	16708	26500	7692	7940	21516	44422	45590
$\mathcal{N}_7$	4627	5458	17830	26962	8731	7945	22637	44998	49070

Table E.11: The number of minimal trades in the 3-$(10, 5, 3)$ designs

initial	final design								
design	$\mathcal{N}_1$	$\mathcal{N}_2$	$\mathcal{N}_3$	$\mathcal{N}_4$	$\mathcal{N}_5$	$\mathcal{N}_6$	$\mathcal{N}_7$	c-trades	total
$\mathcal{N}_1$	1619	5300	23760	34560	2556	9540	36480	24056	25271
$\mathcal{N}_2$	1664	522	54	1248	2124	630	2868	380	380
$\mathcal{N}_3$	1632	598	1362	264	2316	606	1176	306	306
$\mathcal{N}_4$	1466	749	876	1475	2352	624	357	352	370
$\mathcal{N}_5$	1184	3200	8480	12000	3326	10	10560	858	951
$\mathcal{N}_6$	1568	616	1584	1936	2234	149	1600	1562	759
$\mathcal{N}_7$	1518	807	1107	234	2358	639	1048	402	412

Table E.12: Smallest defining set sizes of the 3-$(10, 5, 3)$ designs

design	A_i	N_i	$\log_7 N_i$	$\lvert m_s D \rvert$	$\lvert c_s D \rvert$	$\lvert d_s D \rvert$
$\mathcal{N}_1$	720	5040	4.38	5	4	5
$\mathcal{N}_2$	144	25200	5.21	5	8	8
$\mathcal{N}_3$	16	226800	6.34	6	8	8
$\mathcal{N}_4$	6	604800	6.84	6	8	8
$\mathcal{N}_5$	320	11340	4.80	5	4	6
$\mathcal{N}_6$	64	56700	5.63	5	4	6
$\mathcal{N}_7$	9	403200	6.63	6	8	8

Table E.13: The number of distinct trades in the 2-(10, 5, 4) designs (I)

initial	final design											
design	1	2	3	4	5	6	7	8	9	10	11	12
1	18373	18487	18791	18750	13111	14098	14256	13859	14064	8553	8603	4816
2	18487	18373	18750	18791	13111	14256	14098	14064	13859	8603	8553	4762
3	18679	18685	18957	18945	13305	14313	14628	13610	13776	8458	8648	4783
4	18685	18679	18945	18957	13305	14628	14313	13776	13610	8648	8458	4721
5	18603	18603	18922	18922	12315	14267	14267	13749	13749	8614	8614	4543
6	18405	18497	19044	19320	13315	14537	13844	13852	13542	8744	8379	4623
7	18497	18405	19320	19044	13315	13844	14537	13542	13852	8379	8744	4852
8	18443	18655	17940	18156	12876	14016	13690	14137	13959	8715	8486	4758
9	18655	18443	18156	17940	12876	13690	14016	13959	14137	8486	8715	4869
10	18457	18495	17853	18526	12979	14329	13093	14467	13609	8782	8419	4642
11	18495	18457	18526	17853	12979	13093	14329	13609	14467	8419	8782	5026
12	18508	18132	17568	16684	11838	11828	13444	14424	14404	8308	8916	4974
13	18132	18508	16684	17568	11838	13444	11828	14404	14424	8916	8308	4449
14	18352	18352	17100	17100	11424	13044	13044	14492	14492	8644	8644	4710
15	16684	16684	15388	15388	11878	10576	10576	14299	14299	8275	8275	5140
16	18103	18016	16870	17496	12572	13232	12521	14449	14036	8681	8425	4825
17	18016	18103	17496	16870	12572	12521	13232	14036	14449	8425	8681	5008
18	18676	18676	19165	19165	12345	14192	14192	13940	13940	8578	8578	4521
19	18236	18236	18601	18601	13382	13891	13891	13426	13426	8479	8479	4596
20	18448	18460	19594	19900	13426	14368	14830	13309	13552	8320	8563	4759
21	18460	18448	19900	19594	13426	14830	14368	13552	13309	8563	8320	4594

Table E.14: The number of distinct trades in the 2-(10, 5, 4) designs (II)

initial	final design										
design	13	14	15	16	17	18	19	20	21	c-trades	total
1	4762	5200	2229	7242	7255	10674	8041	7514	7464	35474	36677
2	4816	5200	2229	7255	7242	10674	8041	7464	7514	35474	36677
3	4721	5148	2075	6956	7034	10729	8190	7729	7645	35724	37192
4	4783	5148	2075	7034	6956	10729	8190	7645	7729	35724	37192
5	4543	4966	2007	6965	6965	10095	7799	7825	7825	36066	37103
6	4852	5072	2007	7009	6946	10704	7950	7653	7893	36281	36964
7	4623	5072	2007	6946	7009	10704	7950	7893	7653	36281	36964
8	4869	5272	2330	7391	7325	10574	7978	7173	7260	35479	36082
9	4758	5272	2330	7325	7391	10574	7978	7260	7173	35479	36082
10	5026	5197	2303	7405	7231	10483	7841	7102	7509	36034	36241
11	4642	5197	2303	7231	7405	10483	7841	7509	7102	36034	36241
12	4449	5224	2460	7528	7636	9710	7252	6875	6363	34823	34905
13	4974	5224	2460	7636	7528	9710	7252	6363	6875	34823	34905
14	4710	5091	2416	7620	7620	10070	7308	6899	6899	35196	35253
15	5140	5026	2965	7819	7819	8602	6385	5509	5509	34069	34069
16	5008	5223	2487	7558	7624	10158	7541	6489	6912	35017	35209
17	4825	5223	2487	7624	7558	10158	7541	6912	6489	35017	35209
18	4521	5092	1966	6942	6942	10071	7656	7830	7830	36819	37263
19	4596	4996	1953	6881	6881	10442	7435	7692	7692	36533	36605
20	4594	5209	1930	6766	6859	10933	8206	7739	7834	37888	38060
21	4759	5209	1930	6859	6766	10933	8206	7834	7739	37888	38060

Table E.15: The number of minimal trades in the 2-$(10,5,4)$ designs (I)

initial	final design											
design	1	2	3	4	5	6	7	8	9	10	11	12
1	1415	1272	924	991	1064	1317	1217	942	1006	990	775	638
2	1272	1415	991	924	1064	1217	1317	1006	942	775	990	669
3	1023	1074	1509	1460	1053	1397	960	1365	1191	959	895	676
4	1074	1023	1460	1509	1053	960	1397	1191	1365	895	959	743
5	2061	2061	1991	1991	385	1648	1648	1616	1616	1265	1265	576
6	1687	1585	1362	890	1059	1218	1379	1097	1667	845	1222	790
7	1585	1687	890	1362	1059	1379	1218	1667	1097	1222	845	585
8	739	858	1524	1340	966	1101	1489	1227	997	858	882	636
9	858	739	1340	1524	966	1489	1101	997	1227	882	858	532
10	1656	513	1371	1452	1230	777	1737	1245	1071	937	1248	750
11	513	1656	1452	1371	1230	1737	777	1071	1245	1248	937	489
12	2104	2192	1768	2448	736	1936	1208	1960	1176	1272	736	246
13	2192	2104	2448	1768	736	1208	1936	1176	1960	736	1272	279
14	1440	1440	1880	1880	840	1796	1796	1688	1688	1080	1080	170
15	2196	2196	2826	2826	1998	1638	1638	828	828	660	660	468
16	1484	1252	1832	2360	1234	1102	1586	682	684	314	1208	553
17	1252	1484	2360	1832	1234	1586	1102	684	682	1208	314	464
18	2238	2238	1853	1853	186	1503	1503	1826	1826	1212	1212	568
19	2436	2436	1592	1592	862	462	462	1560	1560	1154	1154	617
20	1449	1476	297	1782	1089	1368	954	1800	1485	1008	945	702
21	1476	1449	1782	297	1089	954	1368	1485	1800	945	1008	801

Table E.16: The number of minimal trades in the 2-$(10,5,4)$ designs (II)

initial	final design										
design	13	14	15	16	17	18	19	20	21	c-trades	total
1	669	736	260	821	815	1321	1192	929	990	437	442
2	638	736	260	815	821	1321	1192	990	929	437	442
3	743	776	286	918	1032	1182	1082	778	1056	482	495
4	676	776	286	1032	918	1182	1082	1056	778	482	495
5	576	737	312	1054	1054	437	1148	1168	1168	953	463
6	585	848	280	818	982	1078	876	1054	925	512	490
7	790	848	280	982	818	1078	876	925	1054	512	490
8	532	746	230	631	693	1421	1111	1023	953	401	401
9	636	746	230	693	631	1421	1111	953	1023	401	401
10	489	723	230	342	1089	1158	1065	921	898	372	372
11	750	723	230	1089	342	1158	1065	898	921	372	372
12	279	250	224	888	520	1202	1040	1128	1252	759	513
13	246	250	224	520	888	1202	1040	1252	1128	759	513
14	170	457	256	760	760	824	1272	1140	1140	544	545
15	468	684	270	27	27	990	486	1242	1242	378	378
16	464	685	159	772	836	1246	883	1028	1186	330	330
17	553	685	159	836	772	1246	883	1186	1028	330	330
18	568	720	292	1048	1048	802	1210	1192	1192	552	497
19	617	900	278	843	843	1166	922	1264	1264	434	418
20	801	900	330	1152	1152	1224	1143	1114	1035	495	496
21	702	900	330	1152	1152	1224	1143	1035	1114	495	496

Table E.17: The blocks, with defining sets, of the 2-(10, 5, 4) designs (I)

1	2	3	4	5	6	7	8	9	10	11
01234	⠇01268	⠃01234	⠃01268	⠁01234	⠃01234	⠇01267	⠃01234	⠇01279	⠇01234	⠇01267
01235	⠇01369	01235	⠇01369	01235	01235	01389	01235	01368	01235	⠇01368
01567	01459	01567	⠄01459	01567	⠄01567	⠃01459	01567	⠃01459	01578	01459
⠃01789	⠃01478	01789	01478	01789	⠃01789	01468	⠃01789	⠃01468	⠃01789	⠃01469
02467	02379	02479	⠃02379	⠃02479	⠃02478	⠄02379	⠇02478	02367	02467	02389
⠇02689	⠇02458	⠃02689	02458	⠅02689	⠇02689	02458	02689	⠄02458	⠄02689	⠇02458
⠃03489	02579	03468	⠃02567	⠇03469	03469	⠇02569	⠄03469	⠃02569	⠃03479	⠄02579
03578	03467	⠇03578	03467	03578	03568	03467	⠇03589	⠇03479	⠇03569	03478
⠃04569	⠄03568	⠄04569	⠇03589	⠅04568	04579	03578	04567	03578	⠇04568	03567
⠄12479	12378	⠇12467	⠄12378	⠆12478	⠃12469	12368	⠄12469	12389	⠄12489	⠇12379
12589	12469	12589	⠇12469	12569	⠃12589	⠃12479	12568	12467	12569	⠃12478
13468	⠃12567	⠃13489	12579	⠇13468	⠇13478	⠃12578	⠃13478	⠄12578	13468	⠃12568
⠇13679	⠇13457	13679	⠃13457	13679	13679	⠇13457	⠄13679	13457	⠃13679	13457
14568	13589	14568	13568	14589	⠄14568	⠃13569	⠃14589	⠇13569	14567	13589
⠃23569	⠃23456	23569	23456	23589	23579	23456	23579	⠃23456	23578	23456
23678	23489	⠄23678	23489	23678	23678	⠄23489	⠃23678	23489	23678	23469
⠇24578	46789	⠇24578	46789	⠂24567	24567	46789	24579	46789	24579	46789
⠄34579	56789	⠃34579	56789	⠂34579	⠄34589	56789	34568	56789	⠃34589	56789

Table E.18: The blocks, with defining sets, of the 2-(10, 5, 4) designs (II)

12	13	14	15	16	17	18	19	20	21
⠆01234	⠁01267	⠇01234	⠃01234	⠃01234	⠇01279	⠇01234	⠄01234	⠇01234	⠇01234
⠅01235	⠅01389	⠃01235	⠇01235	01235	⠃01368	⠁01235	01235	01259	⠃01389
01468	01569	⠇01468	01467	⠇01468	01567	⠄01469	01468	⠃01378	⠃01478
01479	01578	⠃01479	01489	01479	⠃01589	⠂01478	⠄01479	⠃01679	01569
02568	⠁02389	02569	⠃02569	02569	⠇02367	02568	⠃02578	02457	⠇02358
02579	02469	⠄02578	⠃02689	02689	02458	⠄02789	⠃02689	⠄02689	02469
03678	02478	⠄03678	⠄03578	03578	⠃02478	⠁03589	03569	03456	⠄02579
03679	03456	03679	⠃03789	03789	03459	03679	⠃03789	⠃03489	03467
04589	⠂03457	⠃04589	⠄04567	⠄04567	03469	04567	⠄04567	05678	⠄05678
⠁12689	⠅12367	12689	12578	⠃12578	⠄12389	12567	12569	⠃12367	12357
⠁12789	⠄12458	12789	12678	12678	12456	⠁12689	⠃12678	⠄12468	⠃12456
⠇13569	⠂12459	13569	⠃13569	13569	12469	⠁13579	⠃13578	13589	12678
⠃13578	⠇13468	13578	13679	13679	13457	⠃13678	13679	⠃14569	⠃13679
14567	⠃13479	14567	14589	⠇14589	13478	14589	⠄14589	⠄14578	14589
⠂23469	⠂23568	⠃23468	⠃23468	⠃23467	⠃23568	⠂23469	⠃23467	23568	23689
⠇23478	⠇23579	⠄23479	⠃23479	⠃23489	⠇23579	⠆23478	⠃23489	23579	24789
⠂24567	46789	24567	⠄24579	⠃24579	46789	⠄24579	⠄24579	⠇24789	⠄34568
⠅34589	56789	⠃34589	34568	⠄34568	56789	⠂34568	34568	34679	⠃34579

Table E.19: Smallest defining set sizes of the 2-$(10, 5, 4)$ designs

design	A_i	N_i	$\log_{14} N_i$	$\|m_sD\|$	$\|c_sD\|$	$\|d_sD\|$
1 (XV)	1	3628800	5.72	5	7	7
2 (XI)	1	3628800	5.72	5	7	7
3 (XII)	1	3628800	5.72	5	7	7
4 (XIV)	1	3628800	5.72	5	7	7
5 (XVI)	2	1814400	5.46	5	6	6
6 (XIX)	2	1814400	5.46	5	7	7
7 (VI)	2	1814400	5.46	5	7	7
8 (XVII)	2	1814400	5.46	5	7	7
9 (III)	2	1814400	5.46	5	7	7
10 (XX)	6	604800	5.04	5	7	7
11 (V)	6	604800	5.04	5	7	7
12 (VIII)	16	226800	4.67	5	6	7
13 (XVIII)	16	226800	4.67	5	6	7
14 (X)	16	226800	4.67	5	7	7
15 (II)	72	50400	4.10	4	8	8
16 (IV)	8	453600	4.94	4	7	7
17 (I)	8	453600	4.94	4	7	7
18 (IX)	4	907200	5.20	5	6	6
19 (VII)	8	453600	4.94	5	7	7
20 (XIII)	9	403200	4.89	5	7	7
21 (XXI)	9	403200	4.89	5	7	7

Table E.20: The distribution of trades (I)

		2-$(8, 4, 3)$											
		α^*		β^*		γ^*		δ^*		3-$(8, 4, 1)$		2-$(7, 3, 1)$	
m	f	#d	#m	#d	#m	#d	#m	#d	#m	#d	#m	#d	#m
4	6	.	.	.	.	.	.	.	.	.	.	7	7
4	7	17	17	26	26	56	56	14	14	.	.	.	.
4	8	3	3	6	6	.	.	.	.	.	.	.	.
6	7	7	-	14	-	56	-	7	-	.	.	7	-
6	8	21	8	26	-	56	-	21	14	.	.	.	.
7	7	1	-	2	-	8	-	1	-	.	.	1	-
7	8	129	-	146	-	8	-	99	-	.	.	.	.
8	8	199	-	187	-	175	-	217	42	7	7	.	.
9	8	316	-	168	-	.	.	406	-	.	.	.	.
10	8	477	-	334	-	532	-	462	-	.	.	.	.
11	8	232	-	194	-	56	-	266	-	.	.	.	.
12	8	91	-	85	-	91	-	91	-	7	-	.	.
13	8	14	-	14	-	14	-	14	-	.	.	.	.
14	8	1	-	1	-	1	-	1	-	1	-	.	.

Table E.21: The distribution of trades (II)

		2-(10, 4, 2)											
		H_1		H_2		H_3		4-(11, 5, 1)		3-(10, 4, 1)		2-(9, 3, 1)	
m	f	#d	#m	#d	#m	#d	#m	#d	#m	#d	#m	#d	#m
4	8	45	45	21	21	9	9	.	.	.	.	.	.
6	8	.	.	.	.	.	.	.	.	.	.	36	36
6	9	60	-	12	-	4	-	.	.	.	.	.	.
6	10	15	-	3	-	1	-	.	.	.	.	.	.
7	9	.	.	24	-	36	-	.	.	.	.	.	.
7	10	.	.	48	-	72	-	.	.	.	.	.	.
8	8	.	.	.	.	.	.	.	.	.	.	9	-
8	9	.	.	24	-	12	-	.	.	.	.	.	.
8	10	195	-	123	-	79	24	.	.	.	.	.	.
9	9	.	.	.	.	4	-	.	.	.	.	76	-
9	10	.	.	252	-	150	-	.	.	.	.	.	.
10	9	.	.	.	.	.	.	.	.	.	.	54	-
10	10	270	-	486	-	630	72	.	.	.	.	.	.
11	9	.	.	.	.	.	.	.	.	.	.	12	-
11	10	300	-	480	-	790	-	.	.	.	.	.	.
12	9	.	.	.	.	.	.	.	.	.	.	1	-
12	10	455	-	455	-	455	-	.	.	.	.	.	.
13	10	105	-	105	-	105	-	.	.	.	.	.	.
14	10	15	-	15	-	15	-	.	.	.	.	.	.
15	10	1	-	1	-	1	-	.	.	.	.	.	.
16	10	.	.	.	.	.	.	.	.	45	45	.	.
18	9	.	.	.	.	.	.	.	.	10	10	.	.
22	10	.	.	.	.	.	.	.	.	360	360	.	.
24	10	.	.	.	.	.	.	.	.	165	-	.	.
26	10	.	.	.	.	.	.	.	.	765	-	.	.
28	10	.	.	.	.	.	.	.	.	180	-	.	.
30	10	.	.	.	.	.	.	.	.	1	-	.	.
36	10	.	.	.	.	.	.	11	11	.	.	.	.
36	11	.	.	.	.	.	.	55	55	.	.	.	.
48	11	.	.	.	.	.	.	495	495	.	.	.	.
54	11	.	.	.	.	.	.	1540	1320	.	.	.	.
60	11	.	.	.	.	.	.	2079	1584	.	.	.	.
66	11	.	.	.	.	.	.	1	-	.	.	.	.

APPENDIX F

Tables for Chapter 9

TABLE F.1: Results for the $4 \times$ 2-$(6, 3, 4)$ designs

Design	b^*	$\|\text{aut}(D)\|$	$\|m_s D\|$	$\|c_s D\|$	$\|d_s D\|$
P_0, P_0	10	60	3	6	6
P_0, P_1	20	720	0	6	6
P_0, P_2	16	24	5	6	6
P_0, P_3	14	24	4	6	6

TABLE F.2: Results for the $6 \times$ 2-$(6, 3, 6)$ designs

Design	b^*	$\|\text{aut}(D)\|$	$\|m_s D\|$	$\|c_s D\|$	$\|d_s D\|$
P_0, P_0, P_0	10	60	3	9	9
P_0, P_0, P_1	20	60	6	9	9
P_0, P_0, P_2	16	12	7	9	9
P_0, P_0, P_3	14	12	6	8	9
P_0, P_2, P_3	17	6	7	8	9
P_0, P_2, P_4	19	36	3	9	9

TABLE F.3: Results for the $13 \times$ 2-$(6, 3, 8)$ designs

Design	b^*	$\|\text{aut}(D)\|$	$\|m_s D\|$	$\|c_s D\|$	$\|d_s D\|$
P_0, P_0, P_0, P_0	10	60	3	12	12
P_0, P_0, P_0, P_1	20	60	6	12	12
P_0, P_0, P_0, P_2	16	12	7	12	12
P_0, P_0, P_0, P_3	14	12	6	12	12
P_0, P_0, P_1, P_1	20	720	0	12	12
P_0, P_0, P_1, P_2	20	24	8	12	12
P_0, P_0, P_1, P_3	20	24	6	12	12
P_0, P_0, P_2, P_2	16	24	6	12	12
P_0, P_0, P_2, P_3	17	3	9	12	12
P_0, P_0, P_2, P_4	19	6	8	12	12
P_0, P_0, P_3, P_3	14	24	5	12	12
P_0, P_0, P_3, P_6	17	6	9	12	12
P_0, P_2, P_3, P_6	18	8	8	12	12

Table F.4: Results for the 19 × 2-$(6, 3, 10)$ designs

Design	b^*	$\lvert \mathrm{aut}(D) \rvert$	$\lvert m_s D \rvert$	$\lvert c_s D \rvert$	$\lvert d_s D \rvert$
P_0, P_0, P_0, P_0, P_0	10	60	3	15	15
P_0, P_0, P_0, P_0, P_1	20	60	6	15	15
P_0, P_0, P_0, P_0, P_2	16	12	7	15	15
P_0, P_0, P_0, P_0, P_3	14	12	6	15	15
P_0, P_0, P_0, P_1, P_1	20	60	9	15	15
P_0, P_0, P_0, P_1, P_2	20	12	11	15	15
P_0, P_0, P_0, P_1, P_3	20	12	11	15	15
P_0, P_0, P_0, P_2, P_2	16	12	9	15	15
P_0, P_0, P_0, P_2, P_3	17	3	11	15	15
P_0, P_0, P_0, P_2, P_4	19	6	9	15	15
P_0, P_0, P_0, P_3, P_3	14	12	7	14	15
P_0, P_0, P_0, P_3, P_6	17	6	8	15	15
P_0, P_0, P_1, P_2, P_3	20	6	10	15	15
P_0, P_0, P_1, P_2, P_4	20	36	4	15	15
P_0, P_0, P_2, P_2, P_3	17	6	8	15	15
P_0, P_0, P_2, P_2, P_4	19	6	11	15	15
P_0, P_0, P_2, P_3, P_3	17	3	11	14	15
P_0, P_0, P_2, P_3, P_4	19	4	8	15	15
P_0, P_0, P_2, P_3, P_6	18	2	11	14	15

TABLE F.5: Results for the 34×2-$(6, 3, 12)$ designs

Design	b^*	$\lvert\mathrm{aut}(D)\rvert$	$\lvert m_s D\rvert$	$\lvert c_s D\rvert$	$\lvert d_s D\rvert$
P_0, P_0, P_0, P_0, P_0, P_0	10	60	3	18	18
P_0, P_0, P_0, P_0, P_0, P_1	20	60	6	18	18
P_0, P_0, P_0, P_0, P_0, P_2	16	12	7	18	18
P_0, P_0, P_0, P_0, P_0, P_3	14	12	6	18	18
P_0, P_0, P_0, P_0, P_1, P_1	20	60	9	18	18
P_0, P_0, P_0, P_0, P_1, P_2	20	12	12	18	18
P_0, P_0, P_0, P_0, P_1, P_3	20	12	11	18	18
P_0, P_0, P_0, P_0, P_2, P_2	16	12	9	18	18
P_0, P_0, P_0, P_0, P_2, P_3	17	3	11	18	18
P_0, P_0, P_0, P_0, P_2, P_4	19	6	10	18	18
P_0, P_0, P_0, P_0, P_3, P_3	14	12	7	18	18
P_0, P_0, P_0, P_0, P_3, P_6	17	6	8	18	18
P_0, P_0, P_0, P_1, P_1, P_1	20	720	0	18	18
P_0, P_0, P_0, P_1, P_1, P_2	20	24	11	18	18
P_0, P_0, P_0, P_1, P_1, P_3	20	24	8	18	18
P_0, P_0, P_0, P_1, P_2, P_2	20	24	10	18	18
P_0, P_0, P_0, P_1, P_2, P_3	20	3	13	18	18
P_0, P_0, P_0, P_1, P_2, P_4	20	6	11	18	18
P_0, P_0, P_0, P_1, P_3, P_3	20	24	8	18	18
P_0, P_0, P_0, P_1, P_3, P_6	20	6	13	18	18
P_0, P_0, P_0, P_2, P_2, P_2	16	24	7	18	18
P_0, P_0, P_0, P_2, P_2, P_3	17	3	12	18	18
P_0, P_0, P_0, P_2, P_2, P_4	19	3	13	18	18
P_0, P_0, P_0, P_2, P_3, P_3	17	3	11	18	18
P_0, P_0, P_0, P_2, P_3, P_4	19	2	13	18	18
P_0, P_0, P_0, P_2, P_3, P_6	18	2	13	18	18
P_0, P_0, P_0, P_3, P_3, P_3	14	24	5	18	18
P_0, P_0, P_0, P_3, P_3, P_6	17	3	13	18	18
P_0, P_0, P_1, P_2, P_3, P_6	20	8	10	18	18
P_0, P_0, P_2, P_2, P_3, P_3	17	6	10	18	18
P_0, P_0, P_2, P_2, P_3, P_4	19	2	12	18	18
P_0, P_0, P_2, P_2, P_3, P_6	18	4	11	18	18
P_0, P_0, P_2, P_2, P_4, P_4	19	36	5	18	18
P_0, P_0, P_2, P_3, P_3, P_6	18	2	12	18	18

TABLE F.6: Results for the 4×2-$(7,3,2)$ designs

Design	$b^*; n_1, n_2$	$\lvert\mathrm{aut}(D)\rvert$	$\lvert m_s D\rvert$	$\lvert c_s D\rvert$	$\lvert d_s D\rvert$
1.0, 1.0	7; 0,7	168	3	6	6
1.0, 1.1	11; 8,3	48	4	6	6
1.0, 1.2	13; 12,1	24	5	6	6
1.0, 1.7	14; 14,0	42	4	6	6

TABLE F.7: Results for the 10×2-$(7,3,3)$ designs

Design	$b^*; n_1, n_2, n_3$	$\lvert\mathrm{aut}(D)\rvert$	$\lvert m_s D\rvert$	$\lvert c_s D\rvert$	$\lvert d_s D\rvert$
3.0	21; 21,0,0	42	6	7	7
1.0, 1.0, 1.0	7; 0,0,7	168	3	9	9
1.0, 1.0, 1.1	11; 4,4,3	24	5	8	9
1.0, 1.0, 1.2	13; 6,6,1	12	6	9	9
1.0, 1.0, 1.7	14; 7,7,0	21	5	9	9
1.0, 1.1, 1.2	15; 10,4,1	8	6	8	9
1.0, 1.1, 1.6	17; 13,4,0	3	6	8	9
1.0, 1.2, 1.5	19; 18,0,1	144	2	9	9
1.0, 1.2, 1.6	18; 15,3,0	6	6	9	9
1.0, 1.2, 1.10	20; 19,1,0	6	5	8	9

TABLE F.8: Results for the 35 × 2-$(7, 3, 4)$ designs

Design	$b^*; n_1, n_2, n_3, n_4$	$\|\text{aut}(D)\|$	$\|m_sD\|$	$\|c_sD\|$	$\|d_sD\|$
3.0, 1.0	28; 28,0,0,0	168	7	12	12
3.0, 1.1 (+)	25; 22,3,0,0	3	6	12	12
3.0, 1.2	22; 16,6,0,0	3	8	9	10
1.0, 1.0, 1.0, 1.0	7; 0,0,0,7	168	3	12	12
1.0, 1.0, 1.0, 1.1	11; 4,0,4,3	24	5	12	12
1.0, 1.0, 1.0, 1.2	13; 6,0,6,1	12	6	12	12
1.0, 1.0, 1.0, 1.7	14; 7,0,7,0	21	5	12	12
1.0, 1.0, 1.1, 1.1	11; 0,8,0,3	48	4	12	12
1.0, 1.0, 1.1, 1.2	15; 6,6,2,1	4	6	12	12
1.0, 1.0, 1.1, 1.3	15; 8,2,4,1	8	6	12	12
1.0, 1.0, 1.1, 1.6	17; 10,3,4,0	3	7	12	12
1.0, 1.0, 1.1, 1.7	17; 9,5,3,0	3	7	12	12
1.0, 1.0, 1.2, 1.2	13; 0,12,0,1	24	5	12	12
1.0, 1.0, 1.2, 1.5	19; 12,6,0,1	24	7	12	12
1.0, 1.0, 1.2, 1.6	18; 10,6,2,0	2	8	12	12
1.0, 1.0, 1.2, 1.7	17; 7,9,1,0	3	7	12	12
1.0, 1.0, 1.2, 1.10	20; 13,6,1,0	3	7	12	12
1.0, 1.0, 1.7, 1.7	14; 0,0,14,0	42	4	12	12
1.0, 1.0, 1.7, 1.10	20; 12,8,0,0	6	7	12	12
1.0, 1.1, 1.2, 1.3	17; 8,8,0,1	16	6	12	12
1.0, 1.1, 1.2, 1.6	20; 13,6,1,0	1	8	12	12
1.0, 1.1, 1.2, 1.7	19; 12,5,2,0	2	7	12	12
1.0, 1.1, 1.2, 1.8	19; 13,3,3,0	6	7	12	12
1.0, 1.1, 1.2, 1.10	21; 15,5,1,0	2	7	12	12
1.0, 1.1, 1.6, 1.7	20; 12,8,0,0	4	6	12	12
1.0, 1.1, 1.6, 1.11	22; 16,6,0,0	2	7	12	12
1.0, 1.1, 1.6, 1.15	23; 18,5,0,0	6	7	12	12
1.0, 1.1, 1.6, 1.20	23; 18,5,0,0	2	8	12	12
1.0, 1.1, 1.6, 1.21	23; 19,3,1,0	6	6	10	12
1.0, 1.1, 1.6, 1.22	22; 16,6,0,0	2	7	12	12
1.0, 1.1, 1.6, 1.27	20; 12,8,0,0	12	6	12	12
1.0, 1.2, 1.6, 1.8	22; 16,6,0,0	12	6	12	12
1.0, 1.2, 1.6, 1.11	22; 16,6,0,0	4	7	12	12
1.0, 1.2, 1.6, 1.14	22; 16,6,0,0	24	6	12	12
1.0, 1.2, 1.10, 1.13	26; 24,2,0,0	16	4	12	12

TABLE F.9: Results for the 4×3-$(8, 4, 2)$ designs

$b^*; n_1, n_2$	$\lvert\text{aut}(D)\rvert$	$\lvert m_s D\rvert$	$\lvert c_s D\rvert$	$\lvert d_s D\rvert$
14; 0,14	1344	3	6	6
22; 16,6	384	4	6	6
26; 24,2	192	5	6	6
28; 28,0	336	4	6	6

TABLE F.10: Results for the 10×3-$(8, 4, 3)$ designs

$b^*; n_1, n_2, n_3$	$\lvert\text{aut}(D)\rvert$	$\lvert m_s D\rvert$	$\lvert c_s D\rvert$	$\lvert d_s D\rvert$
42; 42,0,0	336	6	7	7
14; 0,0,14	1344	3	9	9
22; 8,8,6	192	5	8	9
26; 12,12,2	96	6	9	9
28; 14,14,0	168	5	9	9
30; 20,8,2	64	6	8	9
34; 26,8,0	24	6	8	9
38; 36,0,2	1152	2	9	9
36; 30,6,0	48	6	9	9
40; 38,2,0	48	5	8	9

TABLE F.11: Results for the 31×3-$(8,4,4)$ designs

Extension	$b^*; n_1, n_2, n_3, n_4$	$\lvert \text{aut}(D) \rvert$	$\lvert m_sD \rvert$	$\lvert c_sD \rvert$	$\lvert d_sD \rvert$
0+	56; 56,0,0,0	1344	7	12	12
1+	50; 44,6,0,0	24	6	12	12
2+	44; 32,12,0,0	24	8	9	10
3+	14; 0,0,0,14	1344	3	12	12
4+	22; 8,0,8,6	192	5	12	12
5+	26; 12,0,12,2	96	6	12	12
6+	28; 14,0,14,0	168	5	12	12
7+	22; 0,16,0,6	384	4	12	12
8+	30; 12,12,4,2	32	6	12	12
9+	30; 16,4,8,2	64	6	12	12
10+	34; 20,6,8,0	24	7	12	12
11+	34; 18,10,6,0	24	7	12	12
12+	26; 0,24,0,2	192	5	12	12
13+	38; 24,12,0,2	192	7	12	12
14+	36; 20,12,4,0	16	8	12	12
15+	34; 14,18,2,0	24	7	12	12
16+	40; 26,12,2,0	24	7	12	12
17+	28; 0,28,0,0	336	4	12	12
18+	40; 24,16,0,0	48	7	12	12
19+	34; 16,16,0,2	128	6	12	12
20+	40; 26,12,2,0	8	8	12	12
21+	38; 24,10,4,0	16	7	12	12
22+	38; 26,6,6,0	48	7	12	12
23+	42; 30,10,2,0	16	7	12	12
24+	40; 24,16,0,0	24	7	12	12
25+	44; 32,12,0,0	8	7	12	12
26+	46; 36,10,0,0	12	8	12	12
27+ = 26+					
28+	46; 38,6,2,0	48	6	10	12
29+ = 25+					
30+ = 24+					
31+	44; 32,12,0,0	24	8	12	12
32+ = 31+					
33+	44; 32,12,0,0	192	6	12	12
34+	52; 48,4,0,0	128	4	12	12

Appendix G

Tables for Chapter 11

Table G.1: Blocks and smallest defining sets for W_1 – W_{10}

W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9	W_{10}
*012	*012	*012	*012	*012	*012	*012	*012	*012	*012
*034	034	*034	034	034	034	*034	*034	*034	*034
056	*056	056	*056	*056	*056	056	056	056	056
*078	*078	078	078	078	078	078	078	078	078
09a	*09a	09a	09a	09a	09a	09a	09a	09a	09a
0bc	0bc	*0bc	*0bc	0bc	*0bc	*0bc	0bc	*0bc	0bc
0de	0de	0de	0de	*0de	0de	0de	*0de	0de	0de
135	135	*135	135	135	135	135	135	135	135
146	146	146	*146	*146	146	*146	*146	146	146
179	179	*179	*179	*179	179	*179	179	*179	*179
*18a	*18a	18a	18a	18a	*18a	18a	*18a	18a	18a
1bd	1bd	*1bd	1bd	1bd	1bd	1bd	1bd	1bd	1bd
*1ce	*1ce	1ce	1ce	1ce	1ce	*1ce	1ce	1ce	*1ce
236	*236	236	236	236	236	236	236	236	*236
*245	245	245	245	245	*245	245	245	*245	*245
27a	27a	*27a	27a	*27a	27a	*27a	27a	27a	27a
289	289	289	*289	289	*289	289	28b	*28b	28b
*2be	*2be	2be	2be	*2be	*2be	2be	29e	29e	*29e
2cd	2cd	2cd	*2cd	2cd	2cd	2cd	2cd	2cd	2cd
37b	37b	37b	*37b	37b	37b	*37b	37c	37c	37c
38c	38c	*38c	38c	*38c	*38c	38d	389	389	389
*39d	*39d	*39d	39d	39e	39e	*39e	*3ad	*3ad	*3ad
*3ae	*3ae	3ae	*3ae	*3ad	*3ad	3ac	*3be	3be	3be
*47c	*47c	47d	*47d	47d	*47d	47e	47e	47e	47e
48b	*48b	*48e	48b	*48e	48e	*48c	48d	*48d	*48d
*49e	49e	49b	*49e	*49c	49c	49b	*49c	49b	49c
4ad	4ad	4ac	4ac	4ab	4ab	4ad	4ab	*4ac	*4ab
*57d	57e	*57e	57e	*57c	57c	57c	*57d	*57d	*57d
58e	58d	58d	*58d	58b	58d	58e	58e	58e	58c
*59b	*59c	*59c	59c	59d	*59b	*59d	*59b	59c	59b
5ac	5ab	5ab	*5ab	5ae	5ae	*5ab	5ac	5ab	5ae
67e	*67d	67c	67c	67e	67e	*67d	*67b	67b	67b
*68d	68e	68b	68e	68d	68b	68b	*68c	68c	*68e
*69c	69b	*69e	69b	*69b	69d	69c	69d	*69d	69d
*6ab	6ac	*6ad	6ad	6ac	*6ac	6ae	6ae	*6ae	6ac

TABLE G.2: Blocks and smallest defining sets for $W_{11} - W_{20}$

W_{11}	W_{12}	W_{13}	W_{14}	W_{15}	W_{16}	W_{17}	W_{18}	W_{19}	W_{20}
*012	012	*012	*012	012	012	*012	012	012	012
034	*034	*034	*034	*034	034	*034	034	*034	034
*056	056	056	056	*056	056	056	056	*056	*056
078	078	*078	078	078	*078	078	078	*078	*078
09a	*09a	09a	*09a	*09a	09a	09a	09a	*09a	09a
0bc	*0bc	0bc	0bc	0bc	0bc	*0bc	*0bc	0bc	0bc
*0de	0de	0de	*0de	*0de	*0de	0de	*0de	0de	0de
*135	*135	135	135	135	135	135	*135	*135	*135
146	146	*146	*146	146	*146	146	*146	146	146
*179	179	179	179	179	*179	*179	*179	179	179
18a	*18a	*18a	18a	18a	18a	18a	*18a	18a	18a
1bd	1bd	1bd	1bd	1bd	1bd	1bd	1bd	*1bd	1bd
1ce	1ce	*1ce	*1ce	1ce	1ce	1ce	1ce	*1ce	*1ce
236	236	236	236	*236	*236	*236	236	236	*236
245	*245	245	245	245	*245	245	*245	245	245
*27a	*27a	27a	*27a	*27b	27b	27b	27b	27b	27b
28b	28b	28b	28b	*28c	28c	28c	*28c	28d	*28d
29e	29e	*29e	29e	29d	29d	29d	29e	29c	29e
2cd	*2cd	*2cd	2cd	2ae	*2ae	*2ae	2ad	2ae	2ac
*37c	37c	37c	37c	37e	37e	37e	37e	37e	*37e
38d	38d	38e	*38e	389	*38d	*38d	389	38b	389
*39b	*39b	39b	39b	3ab	*39c	39c	3ab	39d	*3ab
3ae	3ae	3ad	*3ad	*3cd	3ab	*3ab	*3cd	*3ac	3cd
47e	47e	47e	47d	47d	47a	47a	*47a	47c	47a
48c	48c	48d	*489	48b	489	48b	48d	*489	*48c
49d	49d	49c	4ac	*49e	*4be	*49e	49c	4ad	*49d
4ab	4ab	*4ab	4be	4ac	*4cd	4cd	4be	4be	4be
57d	57b	*57d	57e	57a	*57d	*57d	57d	57a	57d
589	58e	589	58d	58d	58e	*589	58e	58e	58b
5ac	59c	5ac	*59c	59c	59b	5ac	59b	59b	59c
*5be	*5ad	*5be	*5ab	*5be	*5ac	5be	5ac	*5cd	*5ae
67b	*67d	67b	*67b	*67c	67c	67c	67c	67d	67c
*68e	689	68c	68c	68e	*68b	68e	68b	68c	68e
69c	6ac	*69d	69d	69b	69e	69b	69d	*69e	69b
*6ad	*6be	6ae	6ae	6ad	6ad	*6ad	*6ae	6ab	6ad

TABLE G.3: Blocks and smallest defining sets for $W_{21} - W_{30}$

W_{21}	W_{22}	W_{23}	W_{24}	W_{25}	W_{26}	W_{27}	W_{28}	W_{29}	W_{30}
012	*012	*012	*012	*012	*012	012	012	012	012
034	034	034	034	034	*034	034	034	034	*034
*056	*056	056	*056	*056	056	*056	056	056	056
078	078	078	078	078	078	078	078	078	*078
*09a	09a	*09a	09a	*09a	09a	*09a	*09a	*09a	09a
0bc	0bc	*0bc	0bc	0bc	*0bc	0bc	0bc	0bc	0bc
0de	*0de	0de	0de	0de	0de	0de	0de	*0de	*0de
135	135	*135	135	135	135	135	*135	135	*135
146	*146	146	146	146	146	146	146	*146	146
179	179	179	179	179	179	179	179	179	179
*18b	*18b	18a	*18a	18a	*18a	*18a	18a	18a	18a
1ad	1ad	1bd	1bd	1bd	1bd	1bd	*1bd	*1bd	1bd
*1ce	1ce	1ce	*1ce	*1ce	1ce	*1ce	*1ce	1ce	*1ce
*236	236	236	236	236	236	*236	*236	*236	236
*245	245	247	247	247	*247	*247	247	*247	247
27a	27a	25a	25a	*25a	*25b	25b	*25b	25b	25b
*28c	*28c	28b	*28b	*28b	289	28c	28c	*28c	28d
29e	29e	29e	*29e	29e	2ae	29d	29e	29e	29c
2bd	2bd	2cd	2cd	2cd	*2cd	2ae	2ad	2ad	*2ae
37b	37b	37c	37c	*37c	*37c	*37d	37d	37e	37d
*38d	38d	*38d	38d	*38e	38b	389	389	*38b	*389
39c	39c	39b	*39b	*39d	*39e	3ac	*3ac	39d	3ac
3ae	3ae	3ae	*3ae	3ab	3ad	*3be	3be	*3ac	3be
*47c	47e	*45d	*45b	45e	45e	*45a	45a	45a	45a
*48e	*48a	*489	*48e	*48c	48d	48e	48e	489	*48b
49d	*49b	*4ac	49c	49b	*49b	49b	*49b	*4be	49e
4ab	4cd	4be	4ad	4ad	*4ac	4cd	4cd	4cd	4cd
57d	*57c	*57b	57e	*57b	57a	57c	57e	*57d	57c
589	58e	*58e	58c	58d	58c	*58d	*58d	58e	58e
5ac	*59d	59c	*59d	59c	59d	59e	59c	*59c	59d
5be	5ab	67e	*67d	67d	67d	*67e	*67c	67c	*67e
67e	*67d	68c	689	689	*68e	68b	68b	68d	68c
68a	689	69d	6ac	6ac	69c	69c	69d	69b	69b
*69b	6ac	*6ab	6be	6be	6ab	6ad	*6ae	6ae	*6ad
6cd	6be	7ad	7ab	7ae	7be	7ab	7ab	7ab	*7ab

Table G.4: Blocks and smallest defining sets for W_{31} – W_{40}

W_{31}	W_{32}	W_{33}	W_{34}	W_{35}	W_{36}	W_{37}	W_{38}	W_{39}	W_{40}
*012	*012	012	012	*012	012	012	*012	012	*012
*034	034	*034	034	*034	034	*034	*034	034	*034
056	056	*056	*056	056	056	*056	056	056	056
*078	078	078	078	078	078	078	078	078	078
09a	09a	09a	09a	*09a	09a	09a	09a	09a	09a
*0bc	0bc	0bc	*0bc	0bc	0bc	0bc	*0bc	*0bc	0bc
0de	*0de	*0de	*0de	0de	0de	0de	*0de	*0de	0de
135	*135	135	135	135	*135	135	135	*135	135
146	146	*146	146	146	*146	146	146	*146	146
179	179	*179	*179	179	179	*179	*179	*179	*179
18a	*18a	18a	18a	*18a	*18a	*18a	18a	*18a	18a
1bd	1bd	1bd	1bd	*1bd	*1bd	*1bd	*1bd	1bd	1bd
1ce	*1ce	1ce	1ce	1ce	1ce	*1ce	1ce	1ce	*1ce
236	236	236	*236	237	*237	237	237	*237	237
247	247	247	*247	248	248	248	248	248	248
25b	25b	*25b	25b	25b	25b	*25b	*25b	25b	25b
28d	28d	28d	28d	*26c	*26c	*26c	26d	26d	*26d
29c	29c	29e	29e	29d	*29e	*29e	29c	29c	*29c
*2ae	*2ae	*2ac	2ac	2ae	2ad	2ad	2ae	2ae	2ae
37d	37c	*37c	37d	36d	36d	369	*369	*369	36c
*38b	*389	389	38b	*38b	*38b	38d	*38c	38d	*38b
39e	3ad	3ad	*39c	39e	*39c	3ac	*3ad	3ac	39e
3ac	3be	3be	*3ae	3ac	3ae	3be	3be	3be	*3ad
*45a	45a	45a	45a	*45e	45e	*45a	45c	45a	*45e
489	48e	*48e	*48e	*47d	*47c	47e	47e	47b	47c
4be	*49b	*49b	*49b	*49c	49d	49b	49d	49e	49d
*4cd	4cd	4cd	4cd	4ab	*4ab	4cd	4ab	*4cd	*4ab
57c	57d	57e	57e	*57c	57d	57c	57a	*57e	57d
58e	58c	58c	58c	*589	589	58e	58d	*58c	589
*59d	59e	59d	59d	5ad	5ac	59d	59e	59d	5ac
67e	67e	67d	67c	67a	67a	67d	67b	67c	67a
*68c	*68b	68b	689	68e	68e	68b	*68e	68e	68e
69b	*69d	69c	*6ad	69b	69b	6ae	*6ac	6ab	*69b
*6ad	6ac	*6ae	*6be	7be	7be	7ab	7cd	7ad	7be
7ab	*7ab	*7ab	7ab	8cd	8cd	*89c	89b	89b	8cd

TABLE G.5: Blocks and smallest defining sets for W_{41} – W_{50}

W_{41}	W_{42}	W_{43}	W_{44}	W_{45}	W_{46}	W_{47}	W_{48}	W_{49}	W_{50}
012	012	012	*012	*012	012	012	012	012	012
034	034	034	*034	034	034	034	034	*034	*034
*056	056	056	056	056	*056	*056	*056	056	056
078	078	078	078	*078	078	078	078	*078	078
09a	09a	09a	*09a	09a	09a	09a	09a	*09a	09a
0bc	*0bc	*0bc	0bc	0bc	*0bc	*0bc	*0bc	*0bc	0bc
0de	*0de	*0de	*0de	*0de	0de	0de	*0de	0de	*0de
135	*135	*135	135	135	135	135	135	*135	135
146	*146	*146	146	*146	146	146	146	146	146
*179	179	*179	179	179	179	179	*179	179	179
18a	*18a	*18a	*18a	*18a	*18a	*18a	18a	18a	*18a
1bd	1bd	1bd	1bd	1bd	*1bd	1bd	1bd	1bd	1bd
*1ce	1ce	1ce	1ce	1ce	*1ce	1ce	*1ce	*1ce	1ce
237	*237	237	237	*237	*237	*237	*237	237	237
*248	*248	*248	248	248	*248	248	*249	249	*249
*25b	25b	25b	25b	25b	25b	*25b	25b	*25b	25b
*26d	26d	26d	26d	26e	26e	26e	*26e	*26e	26e
29c	29e	29e	29e	29c	29c	29d	28c	28c	*28c
2ae	2ac	2ac	*2ac	2ad	*2ad	2ac	2ad	2ad	2ad
*36a	369	369	36c	*369	369	369	368	36d	*368
38b	38c	38e	38b	38b	*38d	38c	*39d	38e	39b
*39e	*3ad	3ab	39d	3ae	3ac	*3ad	3ac	39b	3ae
3cd	3be	*3cd	3ae	3cd	3be	3be	3be	3ac	3cd
459	45a	*45a	*45e	45e	45a	45a	45a	45c	*45e
47d	47e	47c	*47d	47d	47b	47b	*47b	*47d	*47d
4ac	*49b	49d	49c	*49b	49e	*49e	48e	*48b	48b
*4be	4cd	4be	4ab	*4ac	4cd	4cd	4cd	4ae	4ac
57e	*57d	57e	*57c	57a	*57c	57d	57c	57a	57a
*58c	58e	58d	589	58c	58e	*58e	58d	58d	58d
*5ad	59c	*59c	5ad	59d	59d	*59c	59e	*59e	59c
67c	67c	67b	67a	67c	67d	67c	67d	67c	*67c
68e	*68b	68c	*68e	68d	68c	68d	69c	689	69d
69b	6ae	*6ae	69b	*6ab	6ab	6ab	6ab	6ab	*6ab
7ab	7ab	7ad	*7be	*7be	7ae	*7ae	7ae	7be	*7be
89d	89d	89b	8cd	89e	*89b	*89b	*89b	9cd	89e

Table G.6: Blocks and smallest defining sets for W_{51} – W_{60}

W_{51}	W_{52}	W_{53}	W_{54}	W_{55}	W_{56}	W_{57}	W_{58}	W_{59}	W_{60}
012	012	012	012	012	*012	012	012	012	012
034	034	*034	034	034	034	034	*034	034	034
056	*056	056	056	056	*056	056	056	*056	056
078	078	078	*078	078	078	078	078	078	*078
09a	09a	*09a	*09a	09a	09a	*09a	09a	*09a	*09a
*0bc	*0bc	0bc	0bc	*0bc	0bc	*0bc	0bc	0bc	0bc
*0de	*0de	0de	0de	0de	*0de	0de	0de	*0de	*0de
*135	135	135	135	*135	*135	*135	135	135	*135
*146	146	*146	*146	146	146	146	146	*146	146
*179	179	179	179	*179	179	179	*179	179	*179
*18a	*18a	*18a	18a	18a	*18a	18a	18a	18a	18a
1bd	1bd	1bd	1bd	*1bd	1bd	1bd	1bd	*1bd	1bd
1ce	*1ce	*1ce	*1ce	1ce	1ce	1ce	*1ce	1ce	1ce
*237	237	237	237	237	237	*237	237	237	237
24a	*24a	24a	24a	*24a	24a	24a	*24a	24e	*24e
25b	*25b	*25b	25b	25b	25b	25b	25b	25b	*25b
26e	26e	26e	26e	*26e	*26e	26e	26e	26a	26a
*28c	28c	28c	28c	*28c	28d	28d	28d	28c	*28c
29d	29d	29d	*29d	29d	29c	29c	*29c	*29d	29d
*369	*36c	36b	36b	*368	36d	369	*36b	36c	36d
38d	38d	38d	*38e	39c	*38c	38c	38c	*38b	38e
3ac	*39e	39e	*39c	3ad	39e	*3ad	39e	*39e	39b
3be	3ab	3ac	*3ad	3be	*3ab	3be	*3ad	3ad	*3ac
458	*458	458	459	*459	*458	*458	458	45d	*45c
47e	*47e	47e	*47e	47e	*47e	47e	47e	*47b	47b
49b	*49b	49b	*48b	48b	49b	*49b	49b	489	489
*4cd	4cd	*4cd	4cd	4cd	4cd	4cd	4cd	*4ac	4ad
57d	57d	57d	57c	57c	57c	*57c	*57c	*57a	57a
59c	59c	59c	58d	*58d	59d	59d	*59d	58e	58d
5ae	5ae	*5ae	5ae	5ae	5ae	5ae	5ae	59c	59e
67c	67b	*67c	67d	67d	*67b	*67d	67d	67e	*67e
68b	689	689	689	69b	689	*68b	*689	*68d	68b
6ad	6ad	6ad	*6ac	*6ac	6ac	6ac	6ac	69b	69c
*7ab	7ac	*7ab	7ab	7ab	7ad	7ab	*7ab	7cd	7cd
89e	8be	*8be	9be	89e	8be	*89e	8be	abe	abe

Table G.7: Blocks and smallest defining sets for W_{61} – W_{70}

W_{61}	W_{62}	W_{63}	W_{64}	W_{65}	W_{66}	W_{67}	W_{68}	W_{69}	W_{70}
012	012	012	*012	012	012	012	*012	012	012
034	034	034	034	*034	034	*034	034	034	034
056	056	*056	056	056	056	*056	*056	*056	*056
*078	*078	*078	078	078	078	078	078	078	078
09a	09a	09a	*09a	09a	09a	*09a	09a	09a	*09a
*0bc	0bc	*0bc	0bc	0bc	*0bc	0bc	0bc	*0bc	*0bc
0de	*0de	0de	0de	*0de	0de	0de	0de	*0de	0de
*135	135	135	*135	135	*135	135	135	*135	135
146	*146	*146	146	*146	146	146	146	146	*146
*179	179	179	179	*179	179	179	*179	*179	179
18b	18b	18b	18b	*18b	18b	*18b	18b	18b	18b
1ad	*1ad	*1ad	1ad	1ad	*1ad	*1ad	1ad	1ad	1ad
1ce	1ce	1ce	1ce	1ce	1ce	1ce	1ce	1ce	*1ce
*236	*236	236	236	*237	237	237	237	*237	237
*245	247	247	*247	248	*24b	249	*248	*249	*24d
27a	25a	*25a	*25c	*25d	25d	25c	25a	25b	*25b
28e	28d	28e	*28d	26a	26a	26b	26d	26c	26a
29c	*29c	29c	29b	29c	28c	*28d	*29c	28d	28e
2bd	*2be	*2bd	2ae	*2be	*29e	2ae	2be	2ae	29c
37b	37c	37d	37e	36e	36e	*36e	*36e	36e	36e
*38a	*389	389	38c	389	*389	389	389	*389	389
*39e	3ae	3ac	*39d	3ab	3ab	3ab	*3ab	3ab	*3ac
3cd	3bd	3be	3ab	*3cd	*3cd	3cd	3cd	3cd	3bd
47e	45b	45b	45a	45b	458	45a	45b	45a	45c
*48c	48e	48c	*48e	47e	*47e	47e	47e	47e	47e
49d	49d	*49d	49c	49d	49d	*48c	*49d	48c	48a
4ab	4ac	*4ae	4bd	4ac	4ac	*4bd	4ac	*4bd	49b
57d	*57d	*57c	*57d	57a	57a	*57b	57c	57d	57a
589	58c	58d	589	58c	59c	58e	58d	58e	*58d
5ac	59e	59e	5be	59e	*5be	59d	59e	*59c	59e
*5be	67e	*67e	67b	*67c	67c	67d	*67a	67b	67b
*67c	68a	68a	*68a	68d	*68d	68a	*68c	68a	68c
*68d	*69b	69b	*69e	*69b	*69b	*69c	69b	69d	*69d
69b	6cd	6cd	6cd	7bd	7bd	7ac	7bd	*7ac	7cd
6ae	*7ab	7ab	*7ac	8ae	8ae	9be	*8ae	9be	*abe

Table G.8: Blocks and smallest defining sets for W_{71} – W_{80}

W_{71}	W_{72}	W_{73}	W_{74}	W_{75}	W_{76}	W_{77}	W_{78}	W_{79}	W_{80}
012	012	*012	012	*012	*012	012	012	012	012
*034	*034	*034	034	034	034	034	034	034	*034
056	056	056	*056	*056	*056	056	056	056	*056
078	078	078	078	078	078	078	078	*078	078
09a	09a	09a	09a	*09a	*09a	*09a	09a	09a	09a
0bc	*0bc	0bc	*0bc	0bc	0bc	0bc	*0bc	*0bc	0bc
0de	0de	0de	*0de	0de	0de	*0de	*0de	*0de	*0de
*135	*135	*135	135	135	135	*135	*135	*135	135
146	146	146	146	146	146	*146	146	146	147
179	*179	179	179	*179	*179	179	179	179	168
18b	18b	*18b	18b	18b	*18b	*18b	18b	18b	*19b
*1ad	1ad	1ad	1ad	*1ad	1ad	1ad	1ad	*1ad	1ad
*1ce	1ce	1ce	1ce	1ce	1ce	1ce	*1ce	1ce	1ce
237	237	237	*237	237	*237	237	*237	237	239
*24c	24d	24c	*24a	24c	24d	24d	*24a	249	*246
25a	25c	259	25b	25d	25a	*258	25c	25e	25a
*26b	*26b	26b	26d	*26b	268	26c	*268	26c	27e
*28d	*28a	28d	*28c	28a	29c	*29e	29e	28a	28c
*29e	29e	2ae	29e	29e	*2be	2ab	2bd	2bd	*2bd
36e	36e	36e	36e	*36e	36e	*36e	36e	36a	36b
389	*389	38c	38d	*38d	389	389	*38a	*38e	37c
*3ac	3ac	39d	39b	39b	*3ac	*3ac	39b	39b	38d
3bd	3bd	*3ab	*3ac	3ac	3bd	3bd	3cd	3cd	*3ae
45d	45a	45a	45c	*45a	*45c	45a	459	458	45d
47e	47e	*47e	47e	47e	47b	47c	*47c	47d	489
48a	48c	489	489	489	*48a	48e	48d	*4ac	4ac
*49b	49b	4bd	*4bd	4bd	49e	49b	*4be	4be	4be
57b	57b	57b	57a	57b	57d	57d	57d	*57c	*57b
58e	*58e	58e	*58e	*58e	*58e	59c	*58e	59d	58e
59c	*59d	*5cd	*59d	59c	59b	5be	5ab	*5ab	*59c
67a	*67a	*67d	67b	67a	67c	67b	67b	67b	67a
68c	68d	*68a	*68a	*68c	69d	68a	69d	*68d	69e
*69d	69c	*69c	69c	69d	6ab	*69d	6ac	*69e	*6cd
7cd	7cd	*7ac	7cd	7cd	7ae	*7ae	7ae	7ae	*79d
abe	*abe	9be	abe	abe	*8cd	8cd	89c	89c	*8ab

Table G.9: Sampled $\text{spec}_m(W_i)$ for $W_1 - W_5$

	W_1		W_2		W_3		W_4		W_5	
$\lvert S\rvert$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
12	-	-	-	-	-	-	-	-	-	-
13	-	-	-	-	-	-	4	4	2	2
14	-	-	4	4	30	30	373	323	320	280
15	-	-	137	98	1272	729	5086	1696	4807	1619
16	12	12	2039	657	10931	1282	31082	1321	29895	1309
17	510	345	15012	936	51913	624	111848	350	106750	390
18	5756	564	65559	397	163135	119	286920	40	277429	47
19	35254	960	195914	144	385044	7	580546	3	562641	3
20	133932	111	450890	2	736031	0	982510	0	963503	0
21	366237	341	842254	31	1197309	0	1462905	0	1445859	0
22	768119	1	1348553	0	1726848	0	1974405	0	1960681	0
23	1317122	0	1909763	0	2256409	0	2470146	0	2463417	0

Table G.10: Sampled $\text{spec}_m(W_i)$ for $W_6 - W_{10}$

	W_6		W_7		W_8		W_9		W_{10}	
$\lvert S\rvert$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	-	-	-	-	-	-	-	-	-	-
12	0	0	-	-	0	0	0	0	2	2
13	65	64	193	193	47	44	250	235	190	183
14	2759	1917	5045	2828	2315	1681	6202	3772	6075	3698
15	24924	4019	39511	5601	21644	3533	47593	5594	46054	5622
16	103854	1303	148633	1569	92921	978	173918	1223	171208	1323
17	278629	124	364728	205	255350	77	416026	64	410470	66
18	565965	3	690999	1	531507	4	766991	0	759405	0
19	955330	0	1105465	0	911245	0	1198043	0	1190782	0
20	1411744	0	1571571	0	1367819	0	1667924	0	1659750	0

Table G.11: Sampled $\text{spec}_m(W_i)$ for $W_{11} - W_{15}$

	W_{11}		W_{12}		W_{13}		W_{14}		W_{15}	
$\lvert S\rvert$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	0	0	-	-	-	-	-	-	0	0
12	5	5	0	0	1	1	-	-	5	4
13	435	397	79	73	139	137	58	58	549	489
14	11209	6457	3387	2274	4606	2895	2403	1704	12468	6715
15	80234	9179	30460	5071	36751	4632	22564	3466	84692	8026
16	275130	1862	126696	1486	139907	1067	96324	1102	279680	1380
17	616876	95	335128	145	350891	71	264864	95	605706	66
18	1061291	3	666020	6	674157	1	543535	0	1026439	2
19	1548782	0	1092828	0	1087889	0	924970	0	1490897	0
20	2032739	0	1570662	0	1553598	0	1381182	0	1959778	0

Table G.12: Sampled $\mathrm{spec}_m(W_i)$ for W_{16} – W_{20}

	W_{16}		W_{17}		W_{18}		W_{19}		W_{20}	
$\|S\|$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	-	-	-	-	0	0	0	0	0	0
12	-	-	1	1	3	3	12	10	2	1
13	-	-	488	482	550	500	1073	938	693	631
14	138	138	11890	6476	12635	6921	22595	11643	16026	8762
15	3269	1538	80777	7656	84928	8103	144150	14077	106431	10357
16	23109	1638	268806	1460	278649	1375	450633	2471	345509	1837
17	91345	467	589426	50	603794	54	917735	112	734060	73
18	248813	8	1007827	0	1022026	0	1444571	2	1212447	0
19	526638	0	1474516	0	1486780	0	1955807	0	1717212	0
20	923582	0	1945007	0	1957300	0	2414857	0	2193086	0

Table G.13: Sampled $\mathrm{spec}_m(W_i)$ for W_{21} – W_{25}

	W_{21}		W_{22}		W_{23}		W_{24}		W_{25}	
$\|S\|$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	0	0	0	0	0	0	-	-	0	0
12	4	4	6	6	5	5	1	1	3	2
13	627	548	861	760	1017	920	649	595	720	659
14	13467	7292	18087	9432	21923	11679	15887	9213	16678	8932
15	93228	9982	120028	12352	137974	12772	109927	11743	108512	10430
16	315258	1924	393275	2453	425209	2107	357426	2199	346999	1790
17	691854	102	832819	95	860072	70	761064	70	734349	71
18	1171892	2	1357820	0	1367509	1	1250848	1	1207590	0
19	1682545	0	1886577	0	1868283	0	1756802	0	1708163	0
20	2173295	0	2364990	0	2331224	0	2234500	0	2180795	0

Table G.14: Sampled $\mathrm{spec}_m(W_i)$ for W_{26} – W_{30}

	W_{26}		W_{27}		W_{28}		W_{29}		W_{30}	
$\|S\|$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	-	-	0	0	0	0	-	-	0	0
12	1	1	5	5	13	12	5	5	15	14
13	392	375	1272	1158	1138	1040	847	792	1101	975
14	10627	6238	27851	14580	24277	12934	18241	9854	23605	12592
15	76159	8622	172351	15920	152324	14446	118902	11439	152691	15084
16	264569	1669	520553	2459	466540	2318	376291	1965	478574	2736
17	594884	80	1029335	75	940774	75	783548	72	973687	96
18	1030759	1	1584260	1	1476553	1	1271128	0	1533631	1
19	1513724	0	2104446	0	1997295	0	1774439	0	2065973	0
20	2002847	0	2557853	0	2460582	0	2246362	0	2531959	0

Table G.15: Sampled $\text{spec}_m(W_i)$ for W_{31} – W_{35}

	W_{31}		W_{32}		W_{33}		W_{34}		W_{35}	
$\lvert S\rvert$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	0	0	0	0	-	-	-	-	-	-
12	9	9	13	13	10	10	8	8	7	7
13	1252	1115	1177	1074	1185	1090	1341	1231	1192	1093
14	23833	12467	25401	13682	27563	14887	28603	15008	25342	13286
15	147492	13000	162951	16153	174942	16811	176687	16559	160287	15177
16	449448	1978	503859	2732	535286	2574	531789	2534	494816	2574
17	894271	68	1010989	115	1062818	84	1054675	104	993905	79
18	1401766	0	1574712	0	1640643	3	1624722	2	1553702	1
19	1900897	0	2108740	0	2178103	0	2158736	0	2087454	0
20	2353829	0	2569013	0	2638426	0	2616892	0	2554090	0

Table G.16: Sampled $\text{spec}_m(W_i)$ for W_{36} – W_{40}

	W_{36}		W_{37}		W_{38}		W_{39}		W_{40}	
$\lvert S\rvert$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	1	1	0	0	-	-	0	0	0	0
12	37	31	45	44	18	18	10	9	11	10
13	2276	1883	2927	2480	1688	1523	1231	1111	1279	1133
14	41424	19540	51229	24075	34937	17539	26651	13954	26691	13884
15	239634	19739	289554	23447	210108	19362	168516	16251	168441	15845
16	687644	2834	812376	3422	623214	2970	518616	2638	515511	2575
17	1293239	85	1490573	101	1205401	110	1039028	99	1028140	84
18	1903939	0	2140450	0	1814379	0	1617364	0	1596120	0
19	2430275	0	2671230	0	2358026	0	2158866	0	2132996	0
20	2851931	0	3070963	0	2801082	0	2624525	0	2589778	0

Table G.17: Sampled $\text{spec}_m(W_i)$ for W_{41} – W_{45}

	W_{41}		W_{42}		W_{43}		W_{44}		W_{45}	
$\lvert S\rvert$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	0	0	0	0	0	0	0	0	0	0
12	12	11	22	21	19	16	23	23	12	11
13	1431	1303	1396	1238	1682	1480	2474	2156	1778	1595
14	29532	15157	29017	14787	33997	17349	45614	21951	35233	17963
15	182488	16548	182635	17271	210040	19155	261510	21597	212927	19280
16	552391	2609	560942	2913	625498	3044	742056	3109	629588	2973
17	1085431	109	1117908	119	1212127	109	1379847	76	1214861	93
18	1663592	0	1731731	1	1824181	0	2008451	0	1823059	0
19	2198973	0	2295567	0	2369888	0	2538302	0	2366333	0
20	2651793	0	2764929	0	2809608	0	2951049	0	2810690	0

TABLE G.18: Sampled $\mathrm{spec}_m(W_i)$ for $W_{46} - W_{50}$

	W_{46}		W_{47}		W_{48}		W_{49}		W_{50}	
$\|S\|$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	0	0	0	0	0	0	0	0	1	1
12	20	20	22	19	20	18	29	29	24	23
13	2065	1808	1801	1599	2271	1977	2202	1951	2472	2161
14	40381	19754	36049	18245	42240	20388	42461	20799	45634	21739
15	238774	20988	218367	19820	247522	21181	250784	21645	268693	22946
16	695471	3204	638244	2972	711761	3033	721485	3215	768350	3377
17	1321449	89	1225373	97	1336367	96	1359757	99	1437345	90
18	1955805	0	1831451	0	1964476	1	2000379	1	2093734	0
19	2503042	0	2364049	0	2502353	0	2544162	0	2638537	0
20	2935943	0	2798543	0	2923536	0	2966683	0	3049707	0

TABLE G.19: Sampled $\mathrm{spec}_m(W_i)$ for $W_{51} - W_{55}$

	W_{51}		W_{52}		W_{53}		W_{54}		W_{55}	
$\|S\|$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	0	0	-	-	0	0	0	0	0	0
12	12	11	12	12	12	11	9	9	10	10
13	1486	1358	1796	1638	1753	1543	1227	1124	1674	1533
14	31522	16245	34735	17625	35391	17678	26833	14188	32943	16870
15	192823	17946	209232	19021	212281	18904	171130	16605	198992	18075
16	581761	2875	617891	2898	626406	2897	528070	2791	595234	2789
17	1145370	92	1201008	95	1205811	91	1057258	95	1163478	120
18	1752566	0	1813476	0	1811408	0	1643378	1	1766968	1
19	2304842	0	2358661	0	2350301	0	2189190	0	2315728	0
20	2764667	0	2804548	0	2790389	0	2653915	0	2769265	0

TABLE G.20: Sampled $\mathrm{spec}_m(W_i)$ for $W_{56} - W_{60}$

	W_{56}		W_{57}		W_{58}		W_{59}		W_{60}	
$\|S\|$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	0	0	0	0	0	0	-	-	0	0
12	15	15	45	42	21	21	7	7	14	14
13	2102	1856	3175	2741	2530	2216	846	786	1730	1517
14	38730	19179	55644	25801	45714	21989	19136	10735	35530	17881
15	231586	20449	311745	24973	266217	22124	127105	13662	215986	19778
16	675261	3110	866527	3628	758129	3345	411251	2518	640675	3169
17	1286900	92	1576434	97	1407322	88	867456	98	1242786	102
18	1913330	0	2242200	0	2045327	1	1410855	1	1872759	1
19	2457428	0	2769478	0	2574091	0	1956000	0	2428203	0
20	2890782	0	3155768	0	2982305	0	2452404	0	2876631	0

TABLE G.21: Sampled $\mathrm{spec}_m(W_i)$ for W_{61} – W_{65}

	W_{61}		W_{62}		W_{63}		W_{64}		W_{65}	
$\lvert S\rvert$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	-	-	0	0	0	0	-	-	0	0
12	9	9	34	34	26	24	5	5	17	17
13	1096	983	3134	2787	3019	2641	1162	1068	2075	1872
14	23489	11948	55285	25646	52852	25047	27771	15146	39976	19757
15	149496	14185	309845	24426	299132	24339	180522	18144	238437	21080
16	471666	2458	848694	3303	831541	3515	558319	2940	692680	3181
17	970249	65	1533216	82	1513812	93	1118361	106	1320066	114
18	1535525	1	2173707	0	2158900	0	1725748	1	1955270	0
19	2079480	0	2690052	0	2678681	0	2282794	0	2503870	0
20	2549625	0	3076406	0	3064864	0	2744288	0	2937928	0

TABLE G.22: Sampled $\mathrm{spec}_m(W_i)$ for W_{66} – W_{70}

	W_{66}		W_{67}		W_{68}		W_{69}		W_{70}	
$\lvert S\rvert$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	0	0	0	0	0	0	0	0	0	0
12	15	14	23	23	34	33	38	38	20	19
13	2230	1994	2429	2140	2882	2471	2968	2572	1861	1659
14	44009	21428	45811	22104	51234	24184	51603	24431	36219	18346
15	257342	22239	268976	23341	292346	24219	293713	24418	217626	19621
16	741770	3301	771325	3485	816488	3354	826199	3571	636033	3075
17	1393756	105	1444892	116	1497835	96	1517047	120	1223842	85
18	2041843	0	2107539	2	2148461	0	2175994	0	1832827	1
19	2591189	0	2656400	0	2678768	0	2710935	0	2371369	0
20	3017328	0	3071716	0	3078479	0	3113543	0	2814077	0

TABLE G.23: Sampled $\mathrm{spec}_m(W_i)$ for W_{71} – W_{75}

	W_{71}		W_{72}		W_{73}		W_{74}		W_{75}	
$\lvert S\rvert$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	0	0	0	0	0	0	0	0	0	0
12	31	31	34	33	16	16	25	23	25	25
13	3193	2795	2496	2161	1658	1499	2411	2116	2107	1812
14	55236	26078	45819	22378	32723	17064	45767	22851	41502	20279
15	312691	25596	267689	23184	205193	20057	265926	22554	244576	21153
16	866879	3604	770254	3475	625720	3426	754917	3289	706588	3092
17	1576127	100	1439226	105	1230102	100	1401444	95	1333047	126
18	2240964	1	2104339	0	1871115	0	2034637	0	1965263	1
19	2769171	0	2653491	0	2440826	0	2562492	0	2505380	0
20	3158706	0	3070697	0	2900824	0	2969799	0	2935273	0

TABLE G.24: Sampled $\text{spec}_m(W_i)$ for W_{76} – W_{80}

	W_{76}		W_{77}		W_{78}		W_{79}		W_{80}	
$\|S\|$	#ds	#min	#ds	#min	#ds	#min	#ds	#min	#ds	#min
11	-	-	0	0	0	0	0	0	-	-
12	25	25	43	42	20	20	35	32	33	33
13	2048	1873	3792	3321	2543	2227	2443	2088	3774	3378
14	40343	20301	66801	30942	46122	22207	46116	22844	69222	33656
15	238563	20251	368098	29012	267811	23127	273730	24683	393623	33187
16	683978	2965	999690	4074	766349	3504	793453	3900	1077769	4712
17	1286351	98	1776234	111	1432597	97	1485468	92	1916910	119
18	1892405	0	2470519	1	2084708	0	2150225	0	2650092	1
19	2416782	0	2995152	0	2626649	0	2689069	0	3175825	0
20	2838715	0	3357446	0	3039857	0	3092476	0	3518887	0

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