

Photogrammetry

Lecture 2

Handling rotations, matrixes, and
finding the inner and outer orientation

Overview of Photogrammetry Lectures

- Four Lectures and one workshop session
 - Basic Maths of Photogrammetry
 - – Handling rotations, matrixes, finding the inner and outer orientation
 - Relative and Absolute Orientations, Orthorectification, Block Adjustments, aero triangulation
 - Softcopy Photogrammetry, Photogrammetry and GIS, Satellite Sensors.

Lecture Overview

- Review of Matrix Algebra
- Transformations between coordinate systems
- Collinearity Equations
- Using control points to find the inner and outer orientation

Matrices

- A *Matrix* is an array of numbers in two dimensions
- A *Vector* is an array of numbers in one dimension
- Matrices are convenient ways to represent sets of numbers and operations on them.
- Matrices can represent complex operations very simply and economically.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 12 \\ 12 \\ 5 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 10 \\ 15 \\ 9 \end{bmatrix}$$

↖ Matrices ↗ ↖ Vectors ↗

Matrices are usually represented by a boldface upper case letter. Vectors are usually represented by a boldface lower case letter.

Matrix Addition is achieved by Adding the corresponding elements. Matrices must have the same number of rows and columns.

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{00} + b_{00} & a_{01} + b_{01} & a_{02} + b_{02} \\ a_{10} + b_{10} & a_{11} + b_{11} & a_{12} + b_{12} \\ a_{20} + b_{20} & a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

In scalar multiplication, each matrix element is multiplied by the same quantity

$$s\mathbf{A} = \begin{bmatrix} sa_{00} & sa_{01} & sa_{02} \\ sa_{10} & sa_{11} & sa_{12} \\ sa_{20} & sa_{21} & sa_{22} \end{bmatrix}$$

- In Matrix Multiplication, the number of columns in the left hand matrix must match the number of rows in the right hand matrix. We proceed across the rows of the left hand matrix, and down the columns of the right hand matrix

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_{00}b_{00} + a_{01}b_{10} + a_{02}b_{20} & a_{00}b_{01} + a_{01}b_{11} + a_{02}b_{21} & a_{00}b_{02} + a_{01}b_{12} + a_{02}b_{22} \\ a_{10}b_{00} + a_{11}b_{10} + a_{12}b_{20} & a_{10}b_{01} + a_{11}b_{11} + a_{12}b_{21} & a_{10}b_{02} + a_{11}b_{12} + a_{12}b_{22} \\ a_{20}b_{00} + a_{21}b_{10} + a_{22}b_{20} & a_{20}b_{01} + a_{21}b_{11} + a_{22}b_{21} & a_{20}b_{02} + a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Matrix times a Vector:

$$\mathbf{Ax} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{00}x + a_{01}y + a_{02}z \\ a_{10}x + a_{11}y + a_{12}z \\ a_{20}x + a_{21}y + a_{22}z \end{bmatrix}$$

Matrix Multiplication is not commutative.

$$\mathbf{AB} \neq \mathbf{BA}$$

The Identity Matrix is defined as

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For a square matrix, it may be possible to find an inverse, denoted by the superscript “-1”. Then

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Any Introductory text on Linear Algebra will have details on finding the inverse.

- The transpose of a matrix is obtained by swapping the rows with the columns, and is denoted by a superscript “T”.

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \quad \mathbf{A}^T = \begin{bmatrix} a_{00} & a_{10} & a_{20} \\ a_{01} & a_{11} & a_{21} \\ a_{02} & a_{12} & a_{22} \end{bmatrix}$$

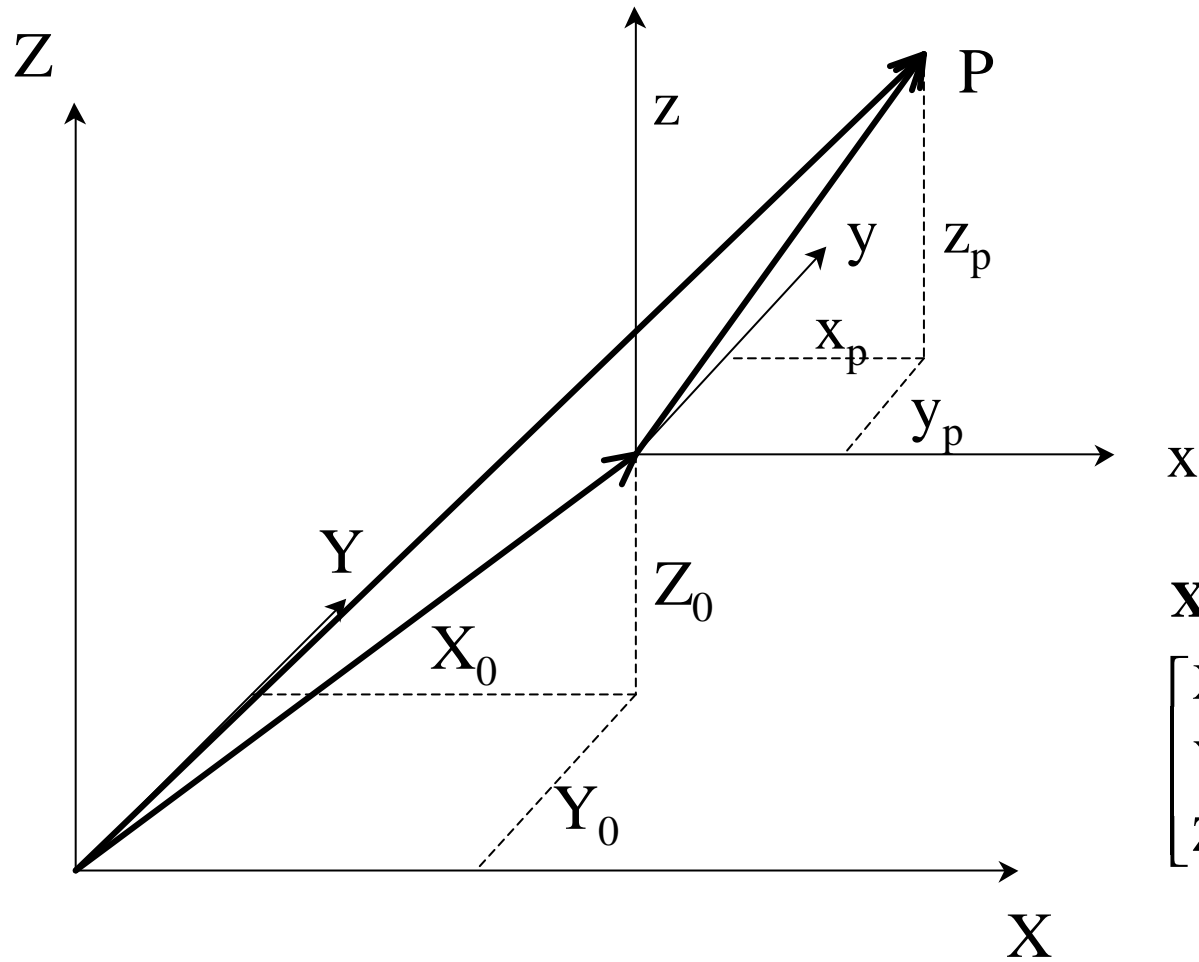
Column 1 becomes Row 1

- We will use matrix algebra to efficiently represent conversions between different coordinate systems.

Coordinate Systems - Review

- Photo Coordinates - origin at centre of collimation.
Denoted by (x,y) in notes.
- Image Coordinate System - origin at centre of Projection.
Denoted by
 - $(x_p, y_p, -f)$ for the image point P'
 - (x_p, y_p, z_p) for the actual point P
 - related by central projection
- World Coordinate System - origin at specified location.
Denoted by (X, Y, Z)
- In the next few slides, we will review how points can be transformed from one coordinate system to another

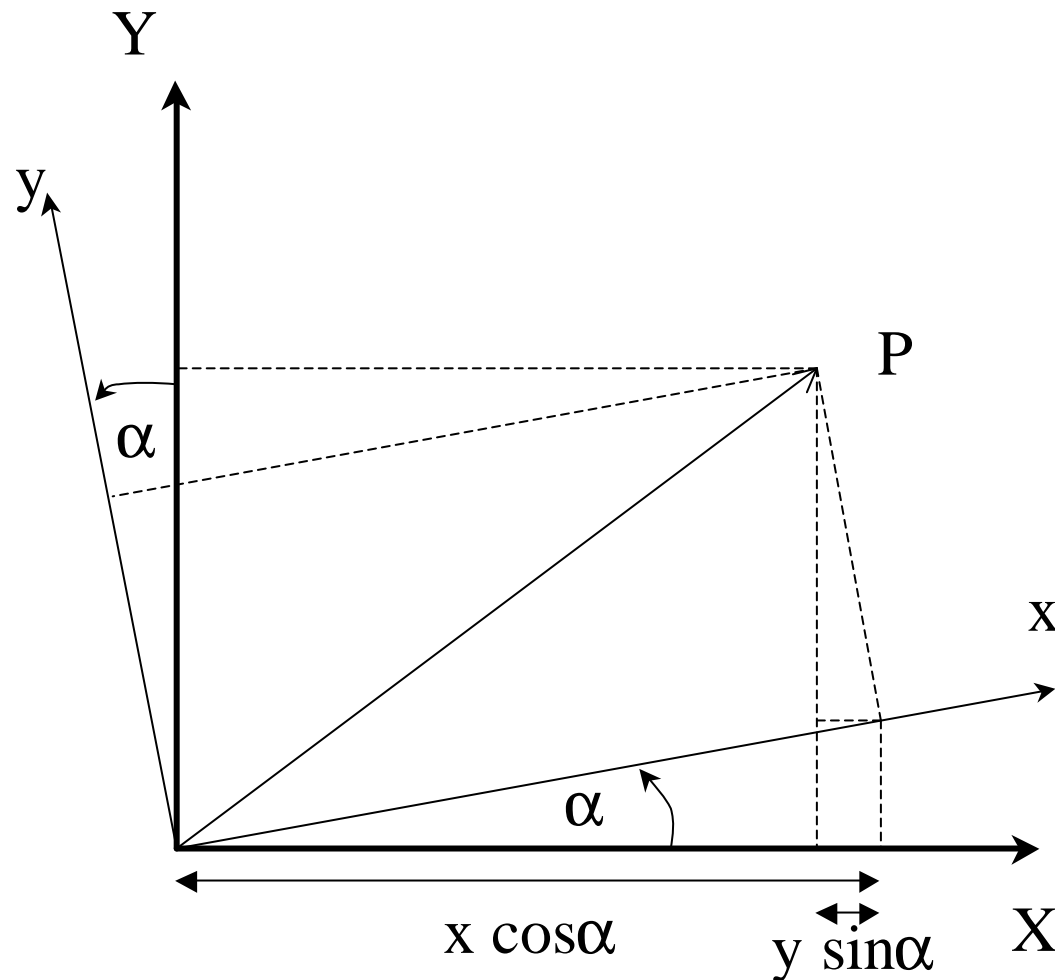
Translation



$$\mathbf{X} = \mathbf{x} + \mathbf{X}_0$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}$$

Rotation - 2D



$$X = x \cos \alpha - y \sin \alpha$$

$$Y = x \sin \alpha + y \cos \alpha$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

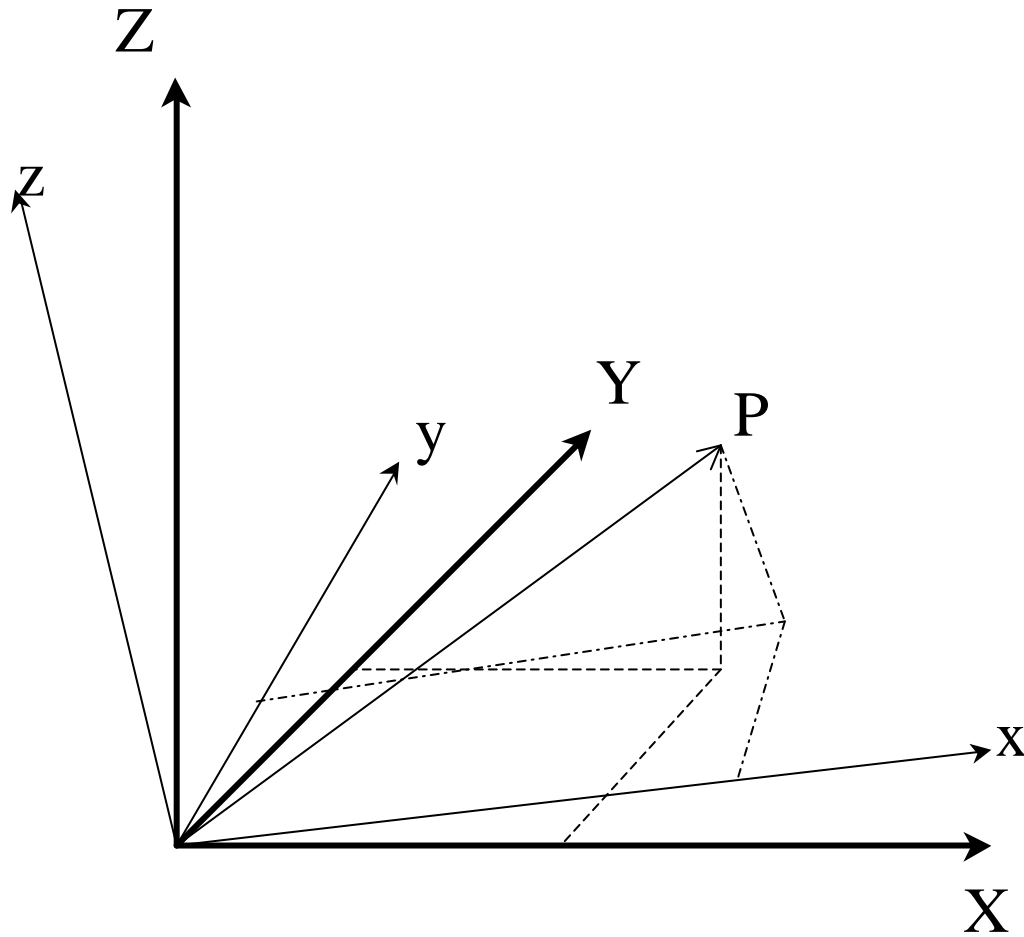
or

$$\mathbf{X} = \mathbf{R}\mathbf{x}$$

where

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

Rotation in 3D



There are a number of systems for representing rotations in three dimensions. They all must be specified with three independent parameters, and can be represented in matrix notation by a 9×9 matrix.

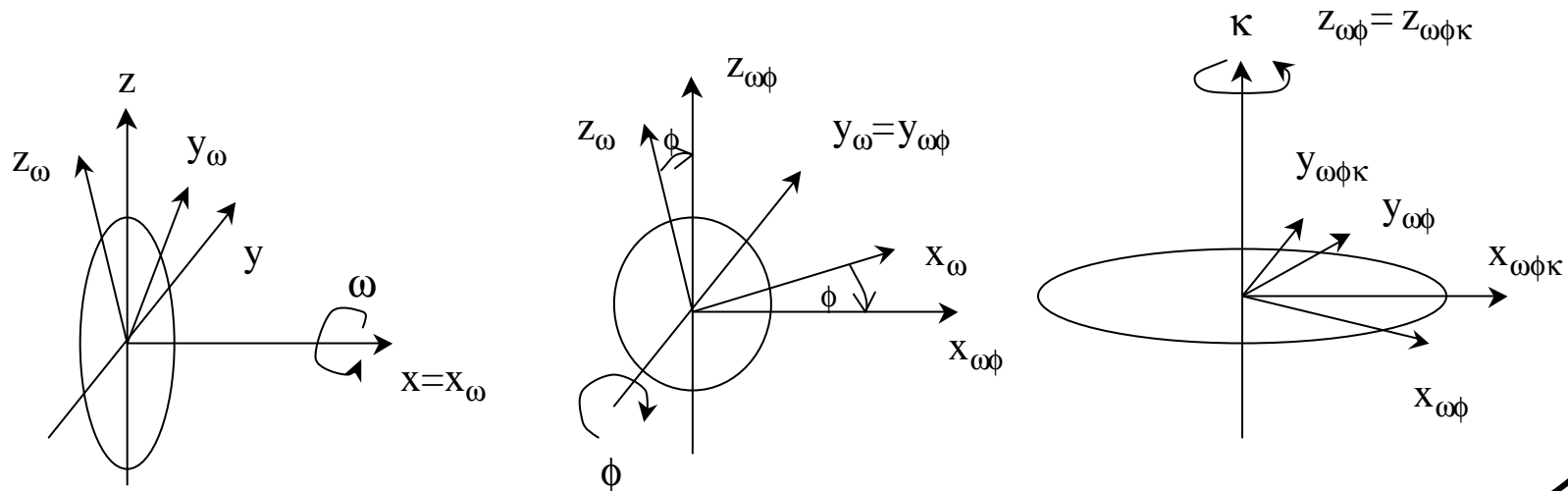
- For any rotation in 3D,

$$\mathbf{X} = \mathbf{R}\mathbf{x} \quad \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- The elements r are functions of the three rotation parameters
- If we let $\cos(xX)$ represent the cosine of the angle between the axes x and X , we can write

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \cos(xX) & \cos(yX) & \cos(zX) \\ \cos(xY) & \cos(yY) & \cos(xZ) \\ \cos(xZ) & \cos(yZ) & \cos(zZ) \end{bmatrix}$$

- In Photogrammetry, a set of three angles of rotations around the three axes is often used to represent a three dimensional rotation.
- ω represents a counterclockwise rotation around the x axis,
- ϕ represents a counterclockwise rotation around the new y axis which was rotated by ω .
- κ represents a counterclockwise rotation around the new z axis, which was rotated by ω , and then by ϕ



- In the ω, ϕ, κ system of specifying rotations, the rotation matrix has the following form

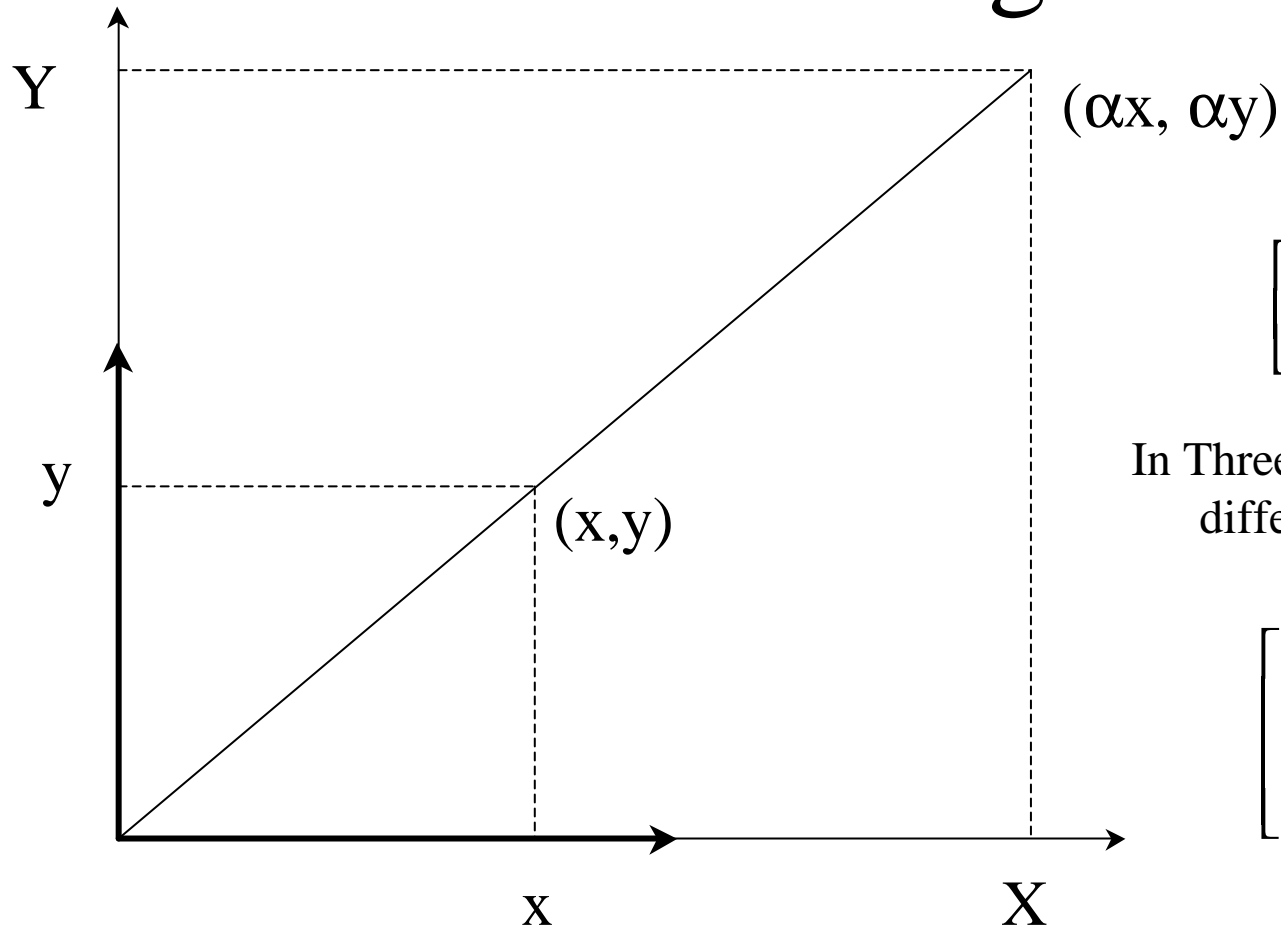
$$\mathbf{R} = \begin{bmatrix} \cos f \cos k & -\cos f \sin k & \sin f \\ \cos w \sin k + \sin w \sin f \cos k & \cos w \cos k - \sin w \sin f \sin k & -\sin w \cos f \\ \sin w \sin k - \cos w \sin f \cos k & \sin w \cos k + \cos w \sin f \sin k & \cos w \cos f \end{bmatrix}$$

- Note that

$$\mathbf{R}^{-1} = \mathbf{R}^T$$

- Note also that the order of the rotations is important. A different overall rotation results from applying the rotational components in a different order.

Scaling



$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{a} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

In Three Dimensions and with
different scale for each
coordinate:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \mathbf{a} & 0 & 0 \\ 0 & \mathbf{b} & 0 \\ 0 & 0 & \mathbf{l} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

General Transformation

- A general transformation between coordinate systems may involve translation, rotation and scaling.

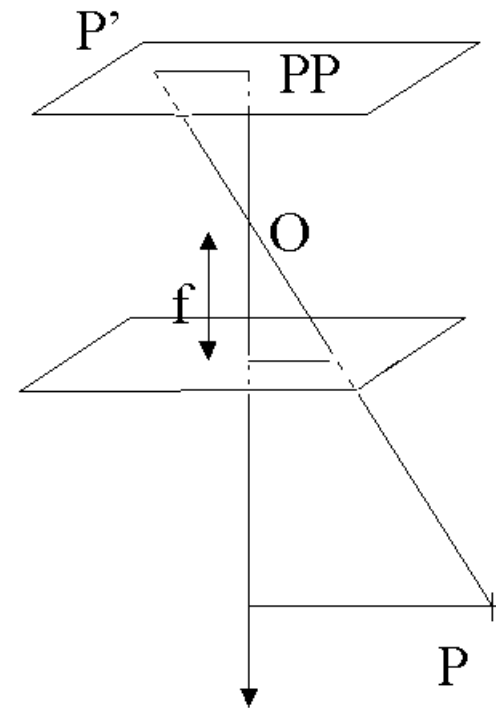
$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \mathbf{a} & 0 & 0 \\ 0 & \mathbf{b} & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}$$

- Such a transformation is a special case of an affine transformation, in which there is a one to one correspondence between points, lines and planes in each coordinate system. In an affine transformation, parallelism is retained. Affine transformations can be represented by

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}$$

Collinearity Equations

- Recall that the basis of locating a real world point P in an image was the fact that the projection centre, the image point, and the real world point lie on a straight line. The mathematical expression of this condition is one of the fundamental relationships of photogrammetry, and is referred to as the “Collinearity Equations”. Having developed a technique for dealing with transformations between coordinate systems, we can now develop the “Collinearity Equations”.

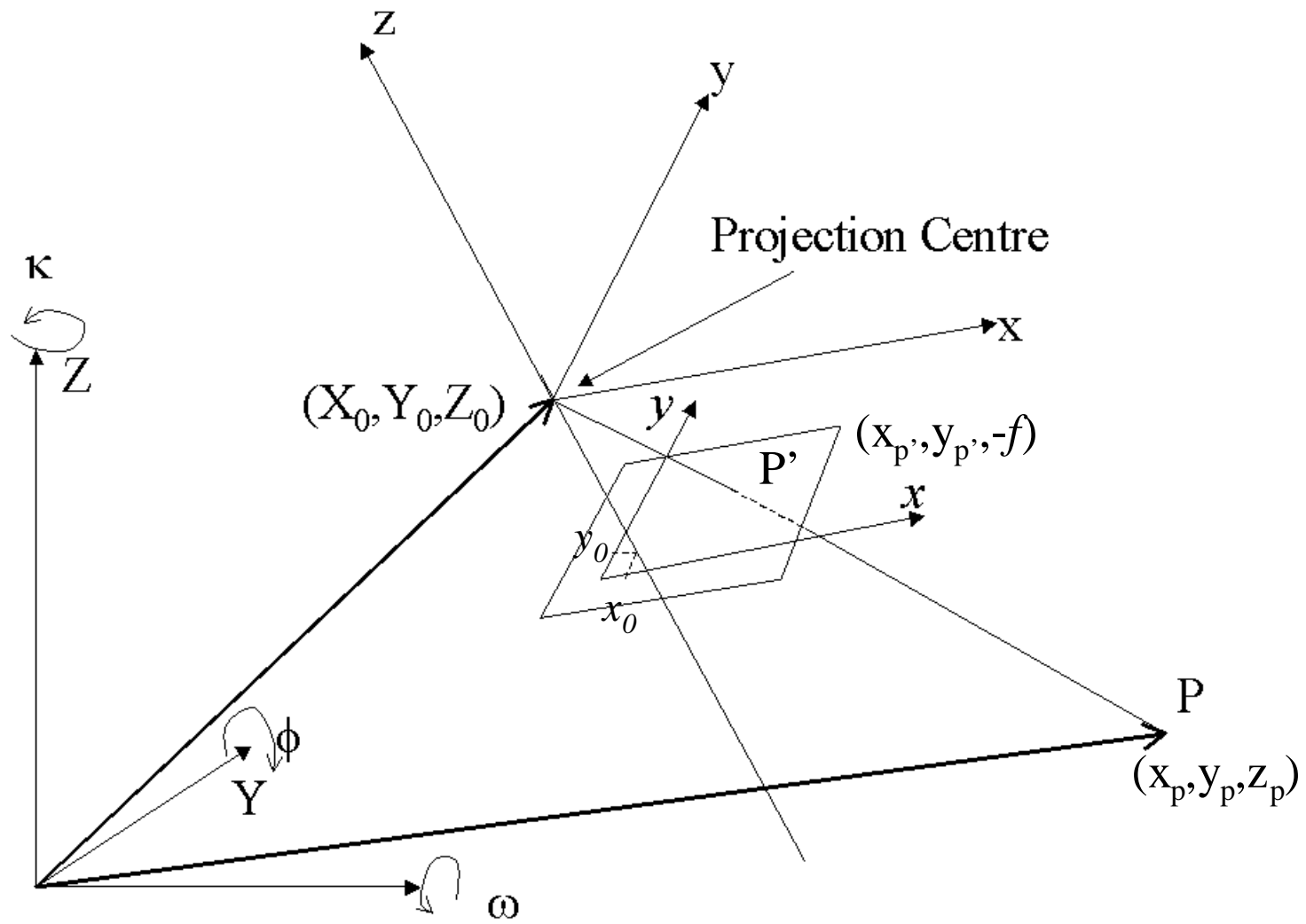


- Let the attitude of the photograph be given in reference to some world coordinate system X, Y, Z , by a rotation matrix R . That is any point $(X, Y, 0)$ would be rotated into a plane parallel to the photograph by the operation of R .
- Assume that the relationship between the image coordinate system and the photo coordinate system is completely specified by a translation.
- The central projection then becomes:

$$\frac{x - x_0}{x} = -\frac{f}{z} \quad \text{and} \quad \frac{y - y_0}{y} = -\frac{f}{z}$$

$$\Leftrightarrow x = x_0 - f \frac{x}{z} \quad \Leftrightarrow y = y_0 - f \frac{y}{z}$$

- Where (x, y) are the photo coordinates, (x_0, y_0) is the location of the principle point, and (x, y, z) is the location of the point in the image coordinate system, with origin at the centre of projection.



- We know that rotation \mathbf{R} specifies the attitude of the photograph, and hence the image coordinate system. Therefore a rotation of \mathbf{R}^{-1} will rotate the image coordinate system to be parallel with world coordinates. Since rotating the coordinate system is equivalent to an inverse rotation of a point, we will need to apply a rotation of \mathbf{R} to the point (x,y,z) to obtain its coordinates in a new coordinate system parallel to the world coordinates.
- Then a simple vector addition can be used to find the world coordinates of the point.

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{R} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}$$

- We can solve this equation for $[x,y,z]^T$, and substitute back into the central projection equations

$$\begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{bmatrix} = \mathbf{R} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{Y}_0 \\ \mathbf{Z}_0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R}^T \begin{bmatrix} \mathbf{X} - \mathbf{X}_0 \\ \mathbf{Y} - \mathbf{Y}_0 \\ \mathbf{Z} - \mathbf{Z}_0 \end{bmatrix}$$

$$x = r_{11}(\mathbf{X} - \mathbf{X}_0) + r_{21}(\mathbf{Y} - \mathbf{Y}_0) + r_{31}(\mathbf{Z} - \mathbf{Z}_0)$$

$$y = r_{12}(\mathbf{X} - \mathbf{X}_0) + r_{22}(\mathbf{Y} - \mathbf{Y}_0) + r_{32}(\mathbf{Z} - \mathbf{Z}_0)$$

$$z = r_{13}(\mathbf{X} - \mathbf{X}_0) + r_{23}(\mathbf{Y} - \mathbf{Y}_0) + r_{33}(\mathbf{Z} - \mathbf{Z}_0)$$

- Substituting into the central projection equations, we get:

$$\begin{aligned} x &= x_0 - f \frac{x}{z} \\ \Leftrightarrow x &= x_0 - f \frac{r_{11}(\mathbf{X} - \mathbf{X}_0) + r_{21}(\mathbf{Y} - \mathbf{Y}_0) + r_{31}(\mathbf{Z} - \mathbf{Z}_0)}{r_{13}(\mathbf{X} - \mathbf{X}_0) + r_{23}(\mathbf{Y} - \mathbf{Y}_0) + r_{33}(\mathbf{Z} - \mathbf{Z}_0)} \end{aligned} \quad (2.1)$$

$$\begin{aligned} \frac{y - y_0}{y} &= -\frac{f}{z} \\ \Leftrightarrow y &= y_0 - f \frac{r_{12}(\mathbf{X} - \mathbf{X}_0) + r_{22}(\mathbf{Y} - \mathbf{Y}_0) + r_{32}(\mathbf{Z} - \mathbf{Z}_0)}{r_{13}(\mathbf{X} - \mathbf{X}_0) + r_{23}(\mathbf{Y} - \mathbf{Y}_0) + r_{33}(\mathbf{Z} - \mathbf{Z}_0)} \end{aligned} \quad (2.2)$$

- Equations (2.1) and (2.2) are known collectively as the collinearity equations, and give the photo coordinates of a point in terms of the
 - World Coordinates of P (X, Y, Z),
 - the exterior orientation parameters
 - the location of the projection centre, (X_0, Y_0, Z_0) and
 - the attitude of the image plane, given by matrix R
 - the interior orientation parameters
 - the location of the principle point, (x_0, y_0)
 - the focal length f
- By using the collinearity equations we can take into account the deviations from normality in aerial photography.
- Improvements are still possible
 - lens distortion, misalignment between image plane and photo-coordinates, atmospheric effects, etc

- The collinearity equations can also be framed to give the world coordinates X and Y as follows

$$X = X_0 + (Z - Z_0) \frac{r_{11}(x - x_0) + r_{12}(y - y_0) - r_{13}f}{r_{31}(x - x_0) + r_{32}(y - y_0) - r_{33}f} \quad (2.3)$$

$$Y = Y_0 + (Z - Z_0) \frac{r_{21}(x - x_0) + r_{22}(y - y_0) - r_{23}f}{r_{31}(x - x_0) + r_{32}(y - y_0) - r_{33}f} \quad (2.4)$$

- If the inner and outer orientation are both known, and points are measured in a pair of overlapping photographs, we obtain (with above ratios denoted by k)

$$X = X_{01} + (Z - Z_{01})k_1 \quad (2.5)$$

$$Y = Y_{01} + (Z - Z_{01})k_2 \quad (2.6)$$

$$X = X_{02} + (Z - Z_{02})k_3 \quad (2.7)$$

$$Y = Y_{02} + (Z - Z_{02})k_4 \quad (2.8)$$

- k_i are functions of photo coordinates, the interior orientation parameters, and the elements of the rotation matrix for each camera.
- Camera One is located at (X_{01}, Y_{01}, Z_{01}) and
- Camera Two is located at (X_{02}, Y_{02}, Z_{02}) .
- Equations (2.5) to (2.8) represent four equations with three unknowns, and so (X, Y, Z) , the location in world coordinates of the imaged point can be computed.
- We have demonstrated how the collinearity equations can be used to perform stereo restitution in the case where inner and outer orientations are known, and the image plane can be oriented arbitrarily towards the world coordinate system. However we have not addressed how to find the inner and outer orientations. This is the subject of the next few sections.

Finding the Orientation Parameters

- The Inner orientation parameters are often known, particularly with metric cameras. They are measured in a laboratory calibration process. However, if they are not known, as is the case when non-metric cameras are used (e.g. video, digital, 35mm), then they can be found. The principle for finding both interior and exterior orientation parameters involves the use of control points.
- Control points are easily identifiable points in the image, whose 3D coordinates are known exactly. The location of the control points can be established by
 - ground survey
 - GPS
 - topographic maps or GIS
- The accuracy of the control points is a primary concern.

- Unfortunately, the set of simultaneous equations is non-linear, and a direct solution is not possible. The usual procedure is to linearize the equations using a Taylor Series Expansion, around an approximate solution. A least squares adjustment is the typical method for solving a set of redundant linear equations.
- If the interior orientation parameters are known, then there are only 6 unknowns. These are the exterior orientation parameters
 - the camera location (X_0, Y_0, Z_0) ,
 - and attitudes ω, ϕ, κ
- In this case a minimum of 3 control points is required.
- The problem of finding the exterior orientation for a single photograph using control points is called **Space Resection**.

Simultaneous orientation of stereo photographs

- This procedure can give a better result than the previous method, because it also allows additional image points to be used. These additional points can be measured in both photographs, but their 3D coordinates are not known.
- For each control point there are four equations, with 12 unknowns (image coordinates are measured)

$$x_{i1} = f(x_0, f, \mathbf{w}_1, \mathbf{f}_1, \mathbf{k}_1, X_{01}, Y_{01}, Z_{01}, X_i, Y_i, Z_i)$$

$$y_{i1} = f(y_0, f, \mathbf{w}_1, \mathbf{f}_1, \mathbf{k}_1, X_{01}, Y_{01}, Z_{01}, X_i, Y_i, Z_i)$$

$$x_{i2} = f(x_0, f, \mathbf{w}_2, \mathbf{f}_2, \mathbf{k}_2, X_{02}, Y_{02}, Z_{02}, X_i, Y_i, Z_i)$$

$$y_{i2} = f(y_0, f, \mathbf{w}_2, \mathbf{f}_2, \mathbf{k}_2, X_{02}, Y_{02}, Z_{02}, X_i, Y_i, Z_i)$$

known

unknown

Known for control
points, but not
additional points

- For each additional point,
 - there are the 12 unknowns corresponding to the exterior orientation parameters.
 - An additional 3 unknowns corresponding to the World Coordinates of the point.
 - Each additional point adds four equations, and three unknowns to the system of equations, adding to the redundancy, and therefore accuracy of the solution.
- This orientation procedure is considered by Krauss (1993) to be the most accurate, since any number of control and additional points can be used.

Two Step Orientation

- So far we have found the exterior orientation parameters in a single step, by solving the collinearity equations for a number of control points.
- By taking more control points, we can increase the redundancy in the system of equations, and achieve a more reliable result. The method involves a numerical solution to the equations, and must be performed on a computer.
- Traditionally, exterior orientation has been performed as a two step process, following a procedure suited to analogue instruments.
 - Relative Orientation
 - Absolute Orientation

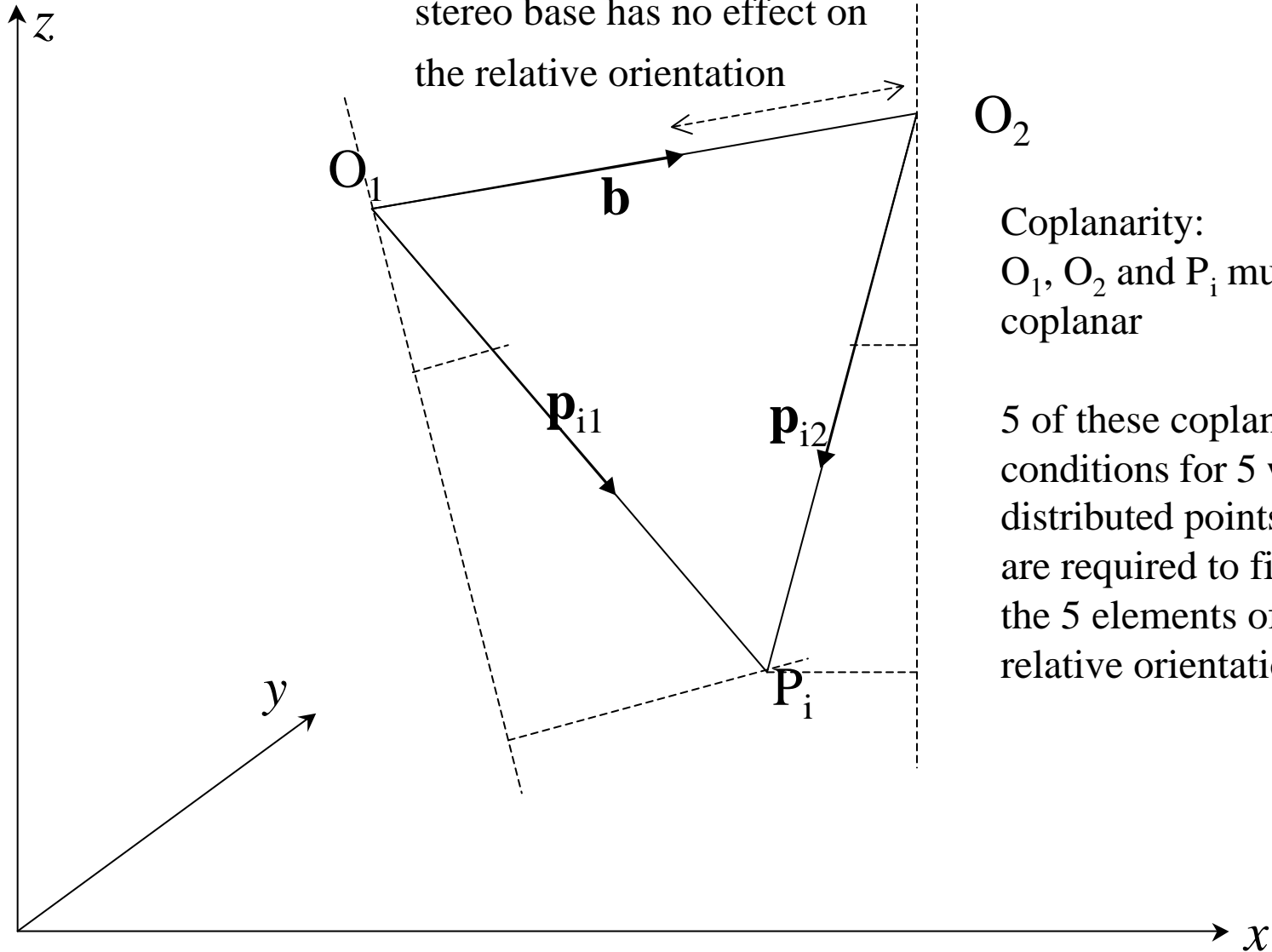
Relative orientation

- Eliminates the y-parallax between the two images of a stereo pair.
- Allows the construction of a 3D **model** coordinate system.
- Points can be precisely located within the model coordinate system.
- It is not known how the model coordinate system is related to the World Coordinate System
- Can be useful if 3D shape information is required, but absolute position or scale is not

Relative Orientation

- Assume the left image is at model coordinates $(X_{01}, Y_{01}, Z_{01}) = (0, 0, 0)$ and ω, ϕ, κ are 0.
- Assume the x coordinate of right image is Camera Two is $X_{02} = b$ (base length)
- In relative orientation we determine the model coordinates Y_{02}, Z_{02} of the right image and its three tilts
- For determining these 5 unknowns we need measurements to 5 or more corresponding image points in both images

Moving O_2 parallel with the stereo base has no effect on the relative orientation



O_2

Coplanarity:
 O_1 , O_2 and P_i must be coplanar

5 of these coplanarity conditions for 5 well distributed points P_i are required to find the 5 elements of relative orientation

Relative orientation

- Achieved by the intersection of five pairs of homologous rays at five well distributed points
- No control points are needed, since no reference is made to world coordinates
- Geometrically, each centre of projection and the model point must be coplanar. (Coplanarity condition)

Absolute orientation

- We now rotate ,shift and scale the model into the world system
- If the model coordinate system is denoted by (x,y,z), and the world coordinate system by (X,Y,Z), then

$$\begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_{\text{mo}} \\ \mathbf{Y}_{\text{mo}} \\ \mathbf{Z}_{\text{mo}} \end{bmatrix} + m \mathbf{R} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- $(\mathbf{X}_{\text{mo}}, \mathbf{Y}_{\text{mo}}, \mathbf{Z}_{\text{mo}})$ is the origin of the model coordinate system
- \mathbf{R} is a rotation matrix
- m is a scaling factor
- Since \mathbf{R} has three independent parameters, there are seven independent elements of absolute orientation.
- Therefore we need at least seven independent equations.
- There are three types of control points
 - Full Control point: (X, Y, Z) all known
 - Three equations

$$\mathbf{X} = \mathbf{X}_{\text{mo}} + m(r_{11}x + r_{12}y + r_{13}z)$$

$$\mathbf{Y} = \mathbf{Y}_{\text{mo}} + m(r_{21}x + r_{22}y + r_{23}z)$$

$$\mathbf{Z} = \mathbf{Z}_{\text{mo}} + m(r_{31}x + r_{32}y + r_{33}z)$$

- Planimetric Control Point: $(X$ and Y known)
 - Two equations, one for X and one for Y

- Height control point: (Z known)
 - One equation for Z

- Seven Equations can be derived from
 - Two full control points and one height control point
 - Three planimetric control points and one height control point
 - Must have at least one height or full control point

- A two step orientation procedure is required for analogue instruments, because the exterior orientation parameters cannot be directly solved, as can be done with a computer.
- In analogue instruments, the second diapositive is adjusted through five degrees of freedom, until y parallax is completely removed.
- This is repeated iteratively until y parallax is zero at the five selected points.
- The absolute orientation is then calculated from control points.
- Because of this historical reason, softcopy photogrammetry systems may still follow the procedure of computing a relative orientation, followed by an absolute orientation.
 - This can be useful if a 3D model is required, but exact world coordinates are not.

Example - Exterior Orientation

- Find the exterior orientation of a photograph, given that
 - the interior orientation is $f = 150\text{mm}$, $x_0 = y_0 = 0$
 - the photography can be considered normal
 - The following control points are available

X(m)	Y(m)	Z(m)	x(mm)	y(mm)
100	100	10	-46.88	-58.59
500	110	50	50.00	-60.00
500	600	60	50.85	63.56
100	550	20	-47.62	47.62

- The collinearity equations are:

$$x = x_0 - f \frac{r_{11}(X - X_0) + r_{21}(Y - Y_0) + r_{31}(Z - Z_0)}{r_{13}(X - X_0) + r_{23}(Y - Y_0) + r_{33}(Z - Z_0)}$$

$$y = y_0 - f \frac{r_{12}(X - X_0) + r_{22}(Y - Y_0) + r_{32}(Z - Z_0)}{r_{13}(X - X_0) + r_{23}(Y - Y_0) + r_{33}(Z - Z_0)}$$

- Since we assume normal photography, there is no rotation, and so \mathbf{R} becomes:

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The collinearity equations reduce to the simple forms:

$$x_i = -f \frac{(X_i - X_0)}{(Z_i - Z_0)} \quad \text{and} \quad y_i = -f \frac{(Y_i - Y_0)}{(Z_i - Z_0)}$$

- Where the subscript i refers to each individual control point.
- In this case, without rotations, there are only three unknowns, so only two of the control points will be required.
- Can solve algebraically:

$$x_i = -f \frac{(X_i - X_0)}{(Z_i - Z_0)}$$

$$(Z_i - Z_0) = -\frac{f(X_i - X_0)}{x_i}$$

$$Z_0 = Z_i + \frac{f(X_i - X_0)}{x_i}$$

- Solving for X_0

$$Z_1 + \frac{f(X_1 - X_0)}{x_1} = Z_2 + \frac{f(X_2 - X_0)}{x_2}$$

$$X_0 = \frac{x_1 x_2 (Z_1 - Z_2)}{f(x_2 - x_1)} + \frac{x_2 X_1 - x_1 X_2}{x_2 - x_1}$$

- Substituting control point information

$$X_0 = \frac{-46.88 \times 50.00 \times (10 - 50)}{150(50.00 - -46.88)} + \frac{50.00 \times 100 - -46.88 \times 500}{50.00 - -46.88}$$

$$= 300\text{m}$$

- Similarly for Y_0

$$Y_0 = \frac{y_1 y_2 (Z_1 - Z_2)}{f(y_2 - y_1)} + \frac{y_2 Y_1 - y_1 Y_2}{y_2 - y_1} = 350\text{m}$$

- Z_0 can then be found from

$$Z_0 = Z_i + \frac{f(X_i - X_0)}{x_i}$$

- If we use $i=1$,

$$Z_0 = 10 + \frac{150 \times (100 - 300)}{-46.88} = 650\text{m}$$

- The other control points can be used as a check. E.g. for $i=3$,

$$Z_0 = 60 + \frac{150 \times (500 - 300)}{50.85} = 650\text{m}$$

- Getting a direct algebraic solution as shown becomes impossible when all the rotations are included as unknowns. This is due to the non-linearity of the resulting system of equations. Iterative numerical methods must then be used.

Summary

- Review of Matrix Algebra
 - Identity, Inverse, Multiplication, addition
- Transformations between coordinate systems
 - Translation, scaling, rotation
- Collinearity Equations
- Using control points to find the inner and outer orientation
 - Simple Example
 - Numerical methods must be used