ENGG7302: Advanced Computational Techniques in Engineering

Lecture 1: Probability

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What will we study in this part of the course?

There will be 8 lectures running over 4 weeks which will cover:

- Probability
- Random variables
- Multiple random variables
- Stochastic process
- Power spectral densities
- Discrete-time stochastic processes
- Markov chains (2 lectures)
- Assessment (class test 10 %, assignment 10 %, part of final exam)

Probability

Probability: Introduction

- What is Probability
  - throw a die, what is the probability that it will show a 4?
  - Definition:
  - Ratio of number of outcomes favourable to A to the no. of possible outcomes

$$P(A) = \frac{N_A}{N}$$

- Trying to calculate the certainity of an event to occur.
- Or Likelihood of an occurence of an event.
- Discrete Probability
  - Countable
- Continuous Probability
  - Infinite

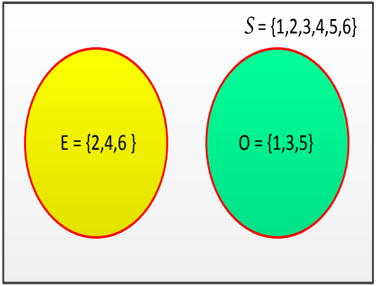
Set theory

- Probability and set theory are closely related. Let us take an example: Rolling of a die.

$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$	Universal Set
Each outcome of $\mathcal{S}$	Element
$E = \{2, 4, 6\}, E \subset \mathcal{S}$	Subset
$O = \{1, 3, 5\}, O \subset \mathcal{S}$	Subset
$E \cap O = \phi$	Disjoint set
$A_{(>6)} = \phi$	Null Set
$A_s = \{1\}$	Simple Set

Set theory

- In short, set diagram or Venn diagram.



Axiomatic approach: Probability

- With the same example now in probability,

$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$	Sample Space
Each outcome of $\mathcal{S}, \mathfrak{s}_i$	Sample point
$E = \{2, 4, 6\}, E \subset \mathcal{S}$	(even) Event $\mathcal{E}$
$O = \{1, 3, 5\}, O \subset \mathcal{S}$	(odd) Event
$E \cap O = \phi$	Mutually Exclusive Events
$A_{(>6)} = \phi$	Impossible Event
$A_s = \{1\}$	Simple Event

- M.E. events: Occurrence of one of the event excludes the other.
- Probability space is the triple of $(\mathcal{S}, \mathcal{E}, P)$, P is the probability measure

Probability

- Suppose we assign a simple event to each outcome of a die say, $E_1 = \{1\}, E_2 = \{2\}, E_3 = \{3\}, \ldots, E_6 = \{6\}$ then,

$$\text{Sample Space} \qquad \mathcal{S} = E_1 \cup E_2 \cup E_3 \cdots \cup E_6$$

$$= \bigcup_{i=1}^{6} E_i$$

$$P[\mathcal{S}] = P[\bigcup_{i=1}^{6} E_i]$$

$$= \sum_{i=1}^{6} P(E_i)$$

- M.E events.

Example

- A box contains m white and n black balls. Balls are drawn at random at a time without replacement. Find the probability of encountering a white ball by the k^{th} draw

$$W_k = \{\text{a white ball is drawn by the } k^{th} \text{ draw }\}$$

$$X_i = \{i \text{ black balls followed by a white ball are drawn}\}$$

$$W_k = X_0 \cup X_1 \cup X_2 \cup \cdots \cup X_{k-1}$$

$$P(W_k) = \sum_{i=0}^{k-1} P(X_i)$$

Example

- Accordingly, lets find the probability for these M.E events.

$$
\begin{aligned}
P(X_0) &= \frac{m}{m+n} \quad W \\
P(X_1) &= \frac{n}{m+n}.\frac{m}{m+n-1} \quad BW \\
P(X_2) &= \frac{n}{m+n}.\frac{n-1}{m+n-1}.\frac{m}{m+n-2} \quad BBW \\
\vdots &= \vdots \\
P(X_{k-1}) &= \frac{n}{m+n}.\frac{n-1}{m+n-1}.\frac{n-2}{m+n-2}\cdots\frac{m}{m+n-(k-1)} \\
P(W_k) &= \sum_{i=0}^{k-1} P(X_i)
\end{aligned}
$$

Conditional Probability

Conditional Probability

- Finding P(A) given the fact that B has taken place.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}$$

$$P(AB) = P(A|B)P(B) \tag{1}$$

- $A \cap B$ is a joint event $P(AB)$ is joint probability.
- De Morgans law applies here as well

$$P(A^c) = 1 - P(A)$$

$$P(A^c|B) = 1 - P(A|B)$$

Conditional Probability

- Now we saw,

$$P(A|B) = \frac{P(AB)}{P(B)}$$

- What would happen if $B \subset A$?

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(B)}{P(B)} \\ &= 1 \end{aligned}$$

- What would happen if $A \subset B$?

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A)}{P(B)} \end{aligned}$$

Example

- A bag contains 5 white and 4 red balls. 2 balls are drawn at random. Find the probability that the first is white and second is red.

$$P(W_1R_2) = P(W_1)P(R_2|W_1)$$
$$= \frac{5}{9}.\frac{4}{8}$$
$$= \frac{5}{18}$$

- Changes: under the conditions of replacement v/s. no replacement.

Example

- A digital communication system transmits one of the three values $-1, 0, 1$. A channel adds noise to cause the decoder sometimes make an error. The error rates are 12.5% if -1 is transmitted, 75% for a 0 transmission and 12.5% if 1 is transmitted. If the probabilites of transmitting various symbols are $P(-1) = P(0) = 0.25$ and $P(1) = 0.5$. Find the probability of error.
- Given Information

$$P(e| - 1) = 0.125, P(e|0) = 0.75, P(e|1) = 0.125$$
$$P(-1) = 0.25, P(0) = 0.25, P(1) = 0.5$$

- Find P(e)

Example

- Probability of error is given as

$$
\begin{aligned}
P(e) &= P(e|-1)P(-1) + P(e|0)P(0) + P(e|1)P(1) \\
&= 0.125 * 0.25 + 0.75 * 0.25 + 0.125 * 0.5 \\
&= 0.4375
\end{aligned}
$$

- HW Comment on your result if $P(-1) = P(0) = P(1)$

Home work

- Suppose box 1 contains a white and b black balls, and box 2 contains c white and d black balls. One ball of unknown colour is transferred from the first box to the second one and then a ball is drawn from the latter. What is the probability that it will be a white ball?

Probability Chain Rule

- Using conditional probability let us now get the chain rule.

$$P(A|B, C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} = \frac{P(ABC)}{P(BC)}$$

- We need an expression for P(ABC), i.e., the joint probability.

$$P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{P(BC)}{P(C)}$$

$$P(ABC) = P(A|B, C)P(B|C)P(C)$$

- So in general, we can say

$$P(A_1, A_2, \ldots, A_n) = P(A_1|A_2, \ldots, A_n) \ldots P(A_{n-1}|A_n)P(A_n)$$

Total Probability Theorem

- Sample Space, $\mathcal{S} = \bigcup_{i=1}^{N} B_i, B_i \cap B_j = \phi (i \neq j)$

$$
\begin{aligned}
P[A] &= P[A \cap \mathcal{S}] \\
&= P[A \cap \bigcup_{i=1}^{N} B_i] \\
&= P[(A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_N)] \\
&= \sum_{i=1}^{N} P[A \cap B_i]
\end{aligned}
$$

- Recall, from conditional probability

$$P(A \cap B) = P(A|B)P(B) \quad \text{we get}$$

$$P[A] = \sum_{i=1}^{N} P[A|B_i]P[B_i]$$

Baye's Rule

- Allows to assess the validity of an event when some other event has taken place. Continuing with conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(AB)}{P(A)}$$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

- Extending further using total probability

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

Statistical Independence

- Two events are independent when $P(A \cap B) = P(AB) = P(A)P(B)$.
- The conditional probability now changes to

$$P(A|B) = \frac{P(A)P(B)}{P(B)} = P(A)$$

- Statistically independent events are different from Mutually Exclusive events.
- Can two events be independent and mutually exclusive?

Independence extended to 3 events

- Continuing with the definition of conditional probability

$$P(A|B) = \frac{P(A)P(B)}{P(B)} = P(A)$$

$$P(A|C) = \frac{P(A)P(C)}{P(C)} = P(A)$$

$$P(A|B) = P(A|C)$$

$$P(A|B,C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} = \frac{P(A)P(B)P(C)}{P(B)P(C)}$$

- Generalising to n independent events, we get

$$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) = P(A_1)P(A_2)P(A_3)\ldots$$
$$\ldots P(A_n)$$

Example

- A fair die is tossed twice. Find the probability of getting a 1 or 2 on the first toss and a 4, 5 or 6 in the second toss. Are these dependent or independent?

$$P(A_1) = \frac{1}{3}, P(A_2) = \frac{1}{2}$$
$$P(A) = P(A_1)P(A_2) = \frac{1}{6}$$

Multiple Trials

Multiple trials

- Repeated coin toss: Succession of sub experiments.
- Total Sample Space would now be given by the **cartesian product**
- $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$
- $\mathcal{S} = \{HH, HT, TH, TT\}$
- Outcome of first $\{H, T\} \in \mathcal{S}_1$and Outcome of second $\{H, T\} \in \mathcal{S}_2$

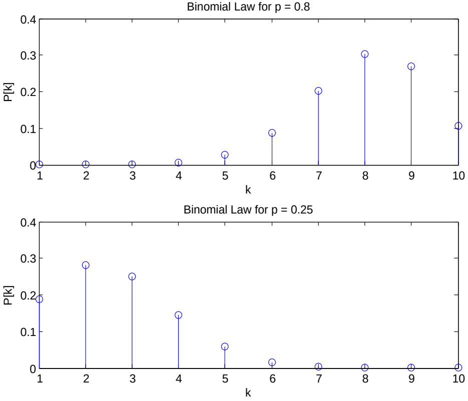
Bernoulli trials and sequence

- An experiment whose outcome is random is Bernoulli trial.
- Interested to know whether a particular event occurs in each, like success or failure.
- $e.g. P[1] = p, P[0] = 1 - p$
- Consecutive independent such trials becomes a sequence.

Binomial Law

- 'M' trials of an experiment are carried out. 'k' successes are observed with probability 'p'. The **Binomial law** is:

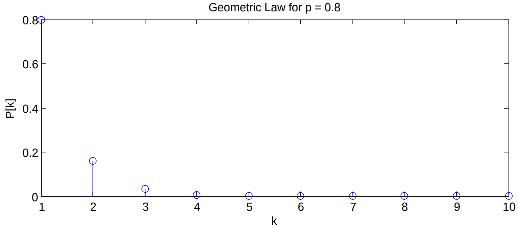
$$P[k] = \binom{M}{k} p^k (1-p)^{M-k} = \frac{M!}{k!(M-k)!} p^k (1-p)^{M-k}$$

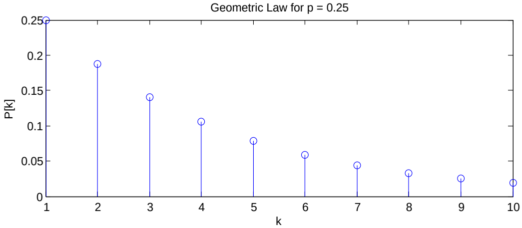


Geometric Law

- **Geometric law** on the other hand observes the first success.

$$p[k] = p(1-p)^{k-1}$$





Example

- A fax machine dials a phone number that is typically busy 80% of the time. The machine dials every 5 mins until the line is clear and the fax is transmitted. What is the probability that the fax machine have to dial the number nine times.
- Probability of success $p = 0.2$. Hence $1 - p = 0.8$. Using the geometric law

$$
\begin{aligned}
P[9] &= (1-p)^{k-1}p \\
&= 0.8^8 * 0.2 \\
&= 0.033.
\end{aligned}
$$

Summary

- Probabilistic questions can be formulated using set theory.
- Conditional probability allows us to assess the certainity of an event A given a known event B has occurred
- If A and B are independent events $P(A \cap B) = P(A)P(B)$.
- Mutually exclusive events are different from independent events.
- The overall sample space of independent subexperiments is given by its cartesian product.