ENGG7302: Advanced Computational Techniques in Engineering

Lecture 2: Random Variables

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Overview of this lecture

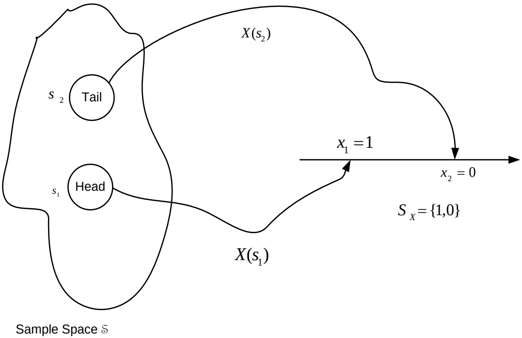
- Discrete random variables
- Continuous random variables
- Distributions
    - Probability mass function
    - Probability Distribution Function
    - Cumulative Distribution Function
- Functions and operations on Random Variable
    - Expectation
    - Variance
    - Std. Deviation
    - Moments and moment generating functions
    - Chebyshev Inequalities

Discrete Random Variables

Discrete Random Variables

- Example: $\mathcal{S}$= { Tail, Head}, $\mathcal{S}_x = \{0, 1\}$
- A function that maps $\mathcal{S}$ into $\mathcal{S}_x$ is called as a Random Variable, denoted by $X(.)$
- $\mathcal{S}_x$ is countable: Discrete Random Variable
- One to one mapping, Many to one mapping
- $\mathcal{S}_x$ is uncountable or infinite: Continuous Random Variable

One to one mapping



- Example: $\mathcal{S}$= { Tail, Head}, $\mathcal{S}_x$ = {1,0}

Probability of Random Variables

- The random variable X(.) maps on an event s_i from a sample space $\mathcal{S}$ to x_i in the sample space $\mathcal{S}_x$
- In one to one mapping its basically the same event with different names.
  Hence, $P[X(s) = x_i] = P[s_j : X(s_j) = x_i] = P\{s_i\}$

Probability of Random Variables

- For many to one mapping

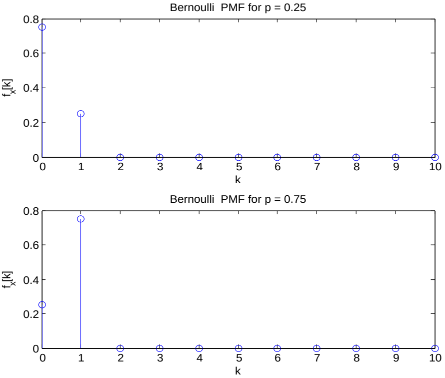
$$
\begin{aligned}
P[X(s) = x_i] &= P[s_j : X(s_j) = x_i] \\
&= \sum_{j : X(s_j) = x_i} P[s_j] \\
f_X[x_i] &= P[X(s) = x_i]
\end{aligned}
$$

- $f_X[x_i]$ is called the Probability Mass Function

Examples of PMF

- Bernoulli PMF

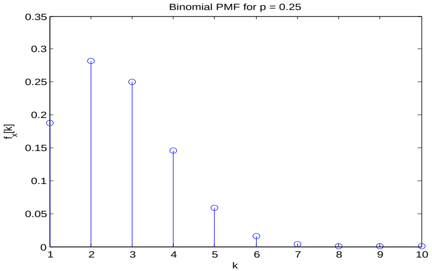
$$
\begin{aligned}
f_X[k] &= 1 - p \qquad k = 0 \\
&= p \qquad\quad\ \ k = 1
\end{aligned}
$$



Examples of PMF

- Binomial PMF

$$f_X[k] = \binom{M}{k} p^k (1-p)^{M-k}$$



- Geometric PMF

$$f_X[k] = p(1-p)^{k-1} \tag{1}$$

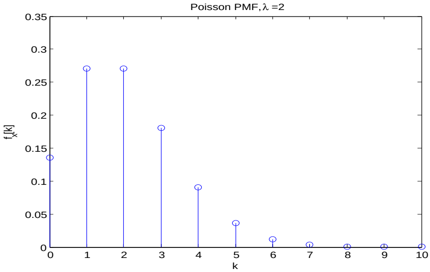
Examples of PMF

- Uniform PMF

$$f_X[k] = \frac{1}{b-a} \qquad a \leq k \leq b$$

- Poisson PMF

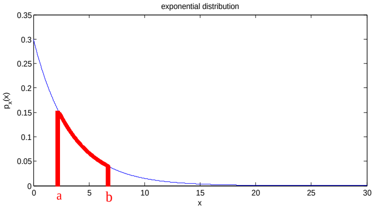
$$f_X[k] = \frac{\lambda^k}{k!}\exp(-\lambda), \quad k = 0, 1, 2, \ldots$$



Continuous Random Variable

Continuous Random Variable

- Example: Length of waiting times at an airport
- $\mathcal{S}_x$ is infinite and uncountable: Continuous R.V.
- Difficult to assign a specific probability to each value of X
- But we can calculate the probability of X lying in an interval.



$$P[a \leq X \leq b] = \int_a^b f_X(x)dx$$

Probability Distribution function (PDF)

- $f_X(x)$ is called as the PDF, $f_X[x_i]$ is called the PMF.
- Properties
    - PDF/ PMF must be non negative

$$f_X(x) \geq 0$$

    - PMF must sum to 1

$$\sum_{i=1}^{M} f_X[x_i] = 1$$

$$\sum_{i=1}^{\infty} f_X[x_i] = 1$$

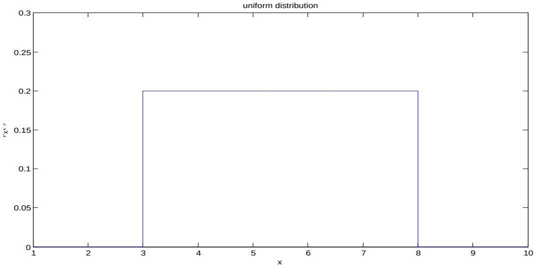
    - PDF must integrate to 1

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

Examples of PDF

- Uniform Distribution, $\mathcal{U}(a, b)$

$$f_x(x) = \frac{1}{b-a} \qquad a \le x \le b$$
$$= 0 \qquad \text{otherwise}$$



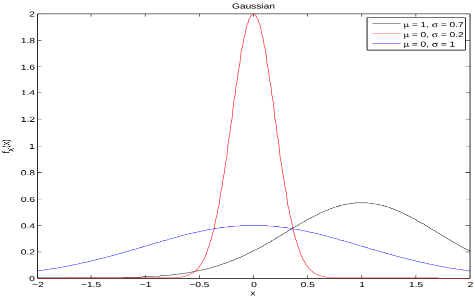
Examples of PDF

- Exponential Distribution

$$f_x(x) = \lambda \exp(-\lambda x) \qquad x \geq 0$$

- Gaussian Distribution

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\} \qquad -\infty < x < \infty$$

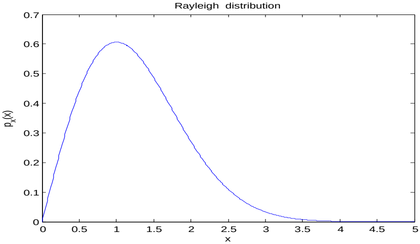


- Gaussian noise ?

Rayleigh

- Rayleigh Distribution

$$f_x(x) = \frac{x}{\sigma^2}\exp\left\{-\frac{1}{2}\frac{x^2}{\sigma^2}\right\} \qquad x \geq 0$$
$$= 0 \qquad x < 0$$



- Rayleigh fading?

Examples of PDF

- Gamma Distribution

$$
\begin{aligned}
f_x(x) &= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\lambda x) \qquad x \geq 0 \\
&= 0 \qquad\qquad\qquad\qquad\qquad\quad x < 0 \\
\Gamma(z) &= \int_0^{\infty} t^{z-1} \exp(-t) dt
\end{aligned}
$$

- Check the family of Matlab functions ‘rand’, ‘randn’, ‘pdf’,‘normpdf’

Cumulative Distribution function (CDF)

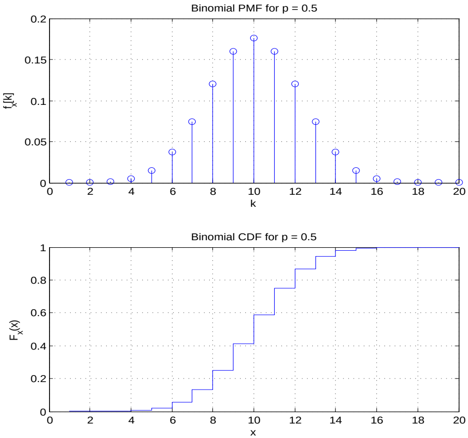
- CDF of a random variable X and evaluated at x

$$F_x(x) = P[X \leq x], \qquad -\infty < x < \infty$$

- Probability that X lies in a semi-infinite interval $(-\infty, x]$
- Discrete RV: CDF can be seen as a **summation** of **PMF's** from $-\infty$ to x
- CDF for discrete variables is a staircase
- Note the representation (for discrete RV)
  - PMF: $f_X[x]$
  - CDF: $F_X(x)$

CDF

- Cumulative distribution function



- Check the family of ‘pdf’, ‘cdf’, ‘binopdf’ functions in Matlab

CDF for continuous RV

- CRV: CDF can be seen as a **integration** of **PDF's** from $-\infty$ to x

$$F_x(x) = \int_{-\infty}^{x} f_x(u)du$$

$$f_x(x) = \frac{dF_x(x)}{dx}$$

- Note the representation (for continuous RV)
  - PDF: $f_X(x)$
  - CDF: $F_X(x)$

Properties of CDF

- CDF is stair case, non decreasing
- $0 \leq F_x(x) \leq 1$
- CDF is right continuous

$$\lim_{x \to -\infty} F_x(x) = 0$$

$$\lim_{x \to \infty} F_x(x) = 1$$

Probability of an interval

- Probability of an interval (a,b) can be found using CDF.

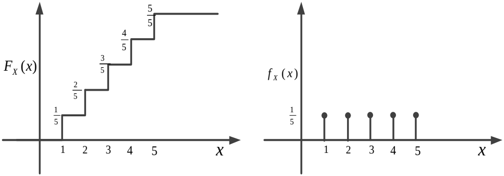
$$\{-\infty < X \leq b\} = \{-\infty < X \leq a\} \cup \{a < X \leq b\}$$
$$P\{-\infty < X \leq b\} = P\{-\infty < X \leq a\} + P\{a < X \leq b\}$$
$$P\{a < X \leq b\} = P\{-\infty < X \leq b\} - P\{-\infty < X \leq a\}$$
$$P[a < X \leq b] = F_X(b) - F_X(a)$$

Examples

- Find the PMF if X is a discrete random variable with the CDF

$$
\begin{aligned}
F_x(x) &= 0 && x < 0 \\
F_x(x) &= \frac{x}{5} && 0 \le x \le 5 \\
&= 1 && x > 5
\end{aligned}
$$

- It is a cumulative distribution function so it will be increasing.



Function of a Random Variable

Function of Random Variable

- If X is a random variable and $g : \mathbb{R} \rightarrow \mathbb{R}$ then $Y = g(X)$ is also a random variable.
- Interested to find the PDF of $Y = g(X)$.
- Suppose we have

$$y = g(x)$$
$$x = g^{-1}(y)$$

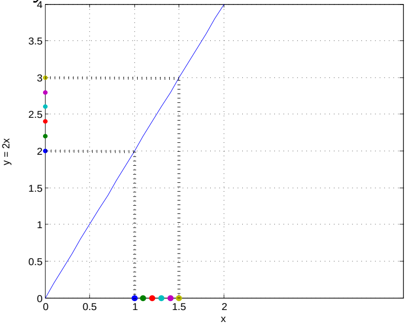
- If the PDF of y is $f_Y(y)$ and PDF of x is $f_X(x)$ we can write $f_X(x)$ as

$$f_X(x) = f_X(g^{-1}(y))$$

- How does this compare to $f_Y(y)$?

Function of Random Variable

- Suppose we have $y = 2x$



- Densities are not distributed properly. So scale it. Hence instead of

$$f_y(y) \approx f_x(g^{-1}(y)) \qquad \text{we have,}$$
$$= \frac{1}{2} f_x(g^{-1}(y))$$
$$= \frac{1}{2} f_x\left(\frac{y}{2}\right)$$

Affine transformation

- Suppose we have $Y = aX + b$, a and b are constants such that $\mathcal{S}_x = \{x : -\infty < x < \infty\}$ and $\mathcal{S}_y = \{y : -\infty < y < \infty\}$
- We have $y = g(x), x = g^{-1}(y)$

$$g(x) = ax + b$$
$$g^{-1}(y) = \frac{y - b}{a}$$
$$\frac{dg^{-1}(y)}{dy} = \frac{1}{a}$$

- The expression for p.d.f of Y can be expressed more generally as

$$f_y(y) = \left|\frac{dg^{-1}(y)}{dy}\right| f_X\left(g^{-1}(y)\right)$$

- This is called as Affine transformation

Operations on Random Variable

Expectation

- Expectation $E[X]$, Mean μ

$$
\begin{aligned}
E[X] &= \sum_i x_i f_X[x_i] \\
&= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= \int_{-\infty}^{\infty} g(x) f_X(x) dx
\end{aligned}
$$

- Expectation is linear

$$E[a_1 X_1 + a_2 X_2] = a_1 E[X_1] + a_2 E[X_2]$$

Variance

- Variance, σ^2

$$
\begin{aligned}
Var(X) &= E[(X-\mu)^2] \\
\sigma^2 &= E[(X-E[X])^2] \\
&= \int_{-\infty}^{\infty} (X-E[X])^2 f_X(x)dx
\end{aligned}
$$

- Variance is a non-linear operation

$$Var(X_1 + X_2) \neq Var(X_1) + Var(X_2)$$

Show that

- Given that $\sigma_X^2 = E[(X-\mu_X)^2]$. Show that $\sigma_X^2 = E[X^2] - \mu_X^2$

$$
\begin{aligned}
\sigma_X^2 &= E[(X-\mu_X)^2] \\
&= E[X^2 + \mu_X^2 - 2X\mu_X] \\
&= E[X^2] + E[\mu_X^2] - E[2X\mu_X] \\
&= E[X^2] + E[\mu_X^2] - 2E[X]E[\mu_X] \\
\text{But} \quad E[\mu_X^2] &= \mu_X^2 \\
&= E[X^2] + \mu_X^2 - 2\mu_X^2 \\
\sigma_X^2 &= E[X^2] - \mu_X^2
\end{aligned}
$$

Example

- Calculate the mean and variance of a Bernoulli random variable.
- The PMF of Bernoulli random variable is defined as:

$$
\begin{aligned}
f_X[x] &= 1 - p \qquad x = 0 \\
&= p \qquad x = 1 \\
&= 0 \qquad \text{otherwise}
\end{aligned}
$$

- The Expectation is given by:

$$
\begin{aligned}
E[X] &= \sum_{k=0}^{1} k f_x[k] \\
&= 0.(1 - p) + 1.p \\
\mu_x &= p
\end{aligned}
$$

Example

- The variance of a Bernoulli random variable is given by:

$$
\begin{aligned}
E[(X-\mu_x)^2] = \sigma_x^2 &= \sum_{k=0}^{1}(k-\mu_x)^2 f_x[k] \\
&= (0-p)^2.(1-p) + (1-p)^2.p \\
\sigma_x^2 &= p^2 - p^3 + p - 2p^2 + p^3 \\
&= p(1-p)
\end{aligned}
$$

Characteristic functions of Random Variable

Moments

- Standard Deviation, $\sqrt{\sigma^2}$
- n^{th} moment is defined as $E[X^n]$

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_x(x)dx.$$

- To determine such n^{th} moments we could also use the characteristic function
- n^{th} central moment is $E[(X - \mu_x)^n]$

$$E[(X - \mu_x)^n] = \int_{-\infty}^{\infty} (x - \mu_x)^n f_x(x)dx$$

- n^{th} absolute moment is $E[|X|^n]$
- n^{th} generalized moment for some a is $E[(X - a)^n]$

Characteristic function of a random variable

- Characteristic function of a random variable is defined as

$$\begin{aligned}\phi(\omega) &= E[\exp(j\omega x)] \\ &= \int_{-\infty}^{\infty} \exp(j\omega x) f_X(x) dx\end{aligned}$$

- Fourier transform of the p.d.f. So the pdf is the IFT of the characteristic function :)

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-j\omega x)\phi(\omega) d\omega$$

- Similarly, for discrete random variables

$$\phi(\omega) = \sum_{k=-\infty}^{\infty} \exp(j\omega k) f_X[k]$$

- DTFT of the PMF $f_X[k]$ and so on.

Using the characteristic function...

$$
\begin{aligned}
\phi(\omega) &= \int_{-\infty}^{\infty} \exp(j\omega x) f_X(x) dx \\
\frac{d\phi(\omega)}{d\omega} &= \int_{-\infty}^{\infty} f_X(x) \frac{d \exp(j\omega x)}{d\omega} dx \\
&= \int_{-\infty}^{\infty} f_X(x) jx \exp(j\omega x) dx \\
\frac{1}{j} \frac{d\phi(\omega)}{d\omega}\bigg|_{\omega=0} &= \int_{-\infty}^{\infty} x f_X(x) dx \\
\frac{1}{j^n} \frac{d^n\phi(\omega)}{d\omega^n}\bigg|_{\omega=0} &= E[X^n]
\end{aligned}
$$

- ...we can obtain the n^{th} moment (Moment generating fn.)

Example

- Use the moment generating function to obtain that mean and variance of an exponential distribution function.

$$
\begin{aligned}
f_X(x) &= \lambda \exp(-\lambda x) && (x \geq 0) \\
&= 0 && (x < 0)
\end{aligned}
$$

The moment generating function is

$$
\begin{aligned}
\phi(\omega) &= \int_0^\infty \lambda \exp -(\lambda - j\omega)x dx \\
&= \frac{\lambda}{\lambda - j\omega}
\end{aligned}
$$

Example: Mean

- Differentiate $\phi(\omega)$ once to get mean and set $\omega = 0$

$$
\begin{aligned}
\frac{d\phi(\omega)}{d\omega} &= \frac{-\lambda(-j)}{(\lambda - j\omega)^2} \\
\frac{1}{j}\frac{d\phi(\omega)}{d\omega}\bigg|_{\omega=0} &= \frac{1}{\lambda} \\
\phi^{'}(0) = E[X] &= \frac{1}{\lambda}
\end{aligned}
$$

Example: Variance

- Differentiate $\phi(\omega)$ twice to get second moment and set $\omega = 0$

$$\frac{d\phi(\omega)}{d\omega} = \frac{\lambda(j)}{(\lambda - j\omega)^2}$$
$$\frac{d^2\phi(\omega)}{d\omega^2} = \frac{-2j\lambda(\lambda - j\omega)(-j)}{(\lambda - j\omega)^4}$$
$$\frac{1}{j^2}\frac{d^2\phi(\omega)}{d\omega^2}\bigg|_{\omega=0} = \frac{2\lambda^2}{\lambda^4}$$
$$\phi^{''}(0) = E[X^2] = \frac{2}{\lambda^2}$$

- Variance

$$\sigma_x^2 = E[X^2] - E[X]^2$$
$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Chebyshev Inequality

- Used to obtain the *partial* information about the probabilites using the mean and variance.
- How do we obtain the probability that a random variable deviates from its mean by more than γ.
- Mathematically, we want to find $P(|X - \mu_x| > \gamma)$

$$
\begin{aligned}
var(X) &= \int_{-\infty}^{\infty} (X - \mu_x)^2 f_X(x) dx \\
&= \int_{x:|x-\mu|>\gamma} (X - \mu_x)^2 f_X(x) dx + \int_{x:|x-\mu|\leq\gamma} (X - \mu_x)^2 f_X(x) dx
\end{aligned}
$$

Chebyshev Inequality

- But we are interested $(|x - \mu_x| > \gamma)$. Therefore,

$$var(X) \geq \int_{x:|x-\mu|>\gamma} (X - \mu_x)^2 f_X(x) dx$$

$$var(X) \geq \int_{x:|x-\mu|>\gamma} \gamma^2 f_X(x) dx$$

$$\frac{var(X)}{\gamma^2} \geq P\Big(|X - \mu_x| > \gamma\Big)$$

- Hence Chebyshev Inequality is

$$P\Big(|X - \mu_x| > \gamma\Big) \leq \frac{var(X)}{\gamma^2}$$