ENGG7302: Advanced Computational Techniques in Engineering

Lecture 3: Multiple Random Variables

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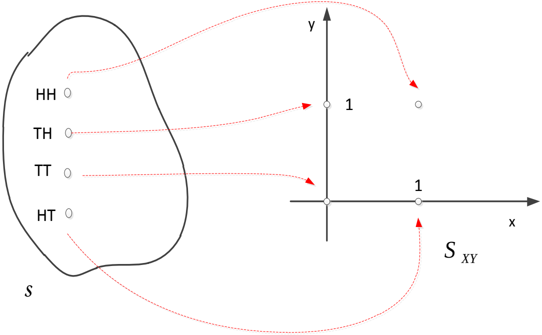
Overview of this lecture

- Concept of Multiple Random Variable
- Joint Probability Distributions
- Marginal Probability Distributions
- Conditional Distributions
- Independent Multiple Random Variables
- Functions of Multiple Random Variable
- Laws and Theorems

Two Random Variables



Concept of Multiple Random Variable

- Outcome of 2 coin tosses $= \{HT, TH, TT, HH\}$
  - To map these events we require 2 random variables.
  - But we are mapping from the same sample space
    $$\mathcal{S} \begin{pmatrix} X(s_i) \\ Y(s_i) \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$
  - Two random variables defined on the same sample space are Jointly distributed.
  - $\mathcal{S}_{X,Y} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$
- These are discrete events, so joint PMF

Joint PMF

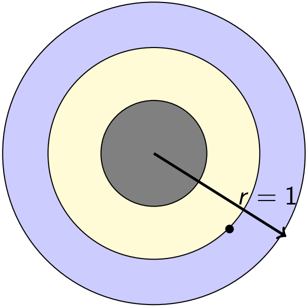
- For a single RV, PMF: $f_X[x_i] = P[X(s) = x_i]$
- For a multiple RV, Joint PMF:

$$f_{X,Y}[x_i, y_j] = P[X(s) = x_i, Y(s) = y_j], i = 1, \ldots, N_x$$
$$j = 1, \ldots, N_y$$

- Properties of Joint PMF

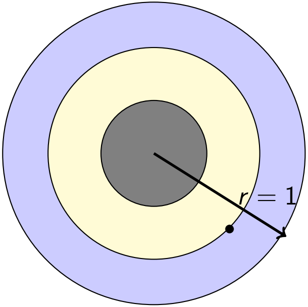
$$0 \leq f_{X,Y}[x_i, y_j] \leq 1$$
$$\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} f_{X,Y}[x_i, y_j] = 1$$

Joint PDF

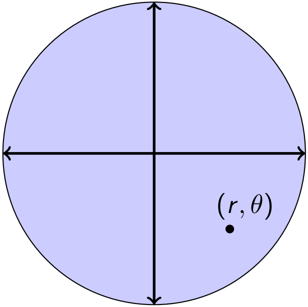


Sample Space $\mathcal{S}$

Joint PDF



Sample Space $\mathcal{S}$



New Sample Space $\mathcal{S}_{r,\theta}$

$$\mathcal{S}_{r,\theta} = \{(r,\theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}$$

Joint CDF

- Joint CDF is given by

$$F_{X,Y}(x,y) = P[X \leq x, Y \leq y]$$
$$= \sum_{i:x_i \leq x} \sum_{j:y_j \leq y} f_{X,Y}[x_i, y_j]$$
$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)dsdt$$

- Properties
  - Monotonically increasing, Right continuous
  - $0 \leq F_{X,Y}(x,y) \leq 1$
  - $f_{X,Y}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$

Marginal Probability Distributions

- Given 2 RV's whose joint distribution is known, the marginal distribution of X is simply the probability distribution of X summing over Y.

$$f_X[x_k] = \sum_{j=1}^{\infty} f_{XY}[x_k, y_j]$$

$$f_Y[y_k] = \sum_{i=1}^{\infty} f_{XY}[x_i, y_k] \qquad \text{Discrete RV, Marginal PMF}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx \qquad \text{Continuous RV, Marginal PDF}$$

Marginal CDF

- Marginal CDF is given by

$$
\begin{aligned}
F_X(x) &= F(x, \infty) \\
&= \lim_{y \to \infty} F_{XY}(x, y) \\
&= F_{XY}(x, \infty) \\
F_y(y) &= F(\infty, y) \\
&= \lim_{x \to \infty} F_{XY}(x, y) \\
&= F_{XY}(\infty, y)
\end{aligned}
$$

Example

- Suppose the joint PMF is given by

$$
\begin{aligned}
f_{X,Y}[i,j] &= 1/8 \qquad & i=0, j=0 \\
&= 1/8 \qquad & i=0, j=1 \\
&= 1/4 \qquad & i=1, j=0 \\
&= 1/2 \qquad & i=1, j=1.
\end{aligned}
$$

- Obtain marginal PMF's $f_X[0]$, $f_Y[1]$.

Example

- Suppose the joint PMF is given by

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&= 1/8 \qquad & i=0, j=1 \\
&= 1/4 \qquad & i=1, j=0 \\
&= 1/2 \qquad & i=1, j=1.
\end{aligned}
$$

- Obtain marginal PMF's $f_X[0]$, $f_Y[1]$.

$$
\begin{aligned}
f_X[0] &= \sum_{j=0}^{1} f_{XY}[i,j] \\
&= f_{XY}[0,0] + f_{XY}[0,1] \qquad &= \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \\
f_Y[1] &= \sum_{i=0}^{1} f_{XY}[i,j] \\
&= f_{XY}[0,1] + f_{XY}[1,1] \qquad &= \frac{1}{8} + \frac{1}{2} = \frac{5}{8}
\end{aligned}
$$

- Joint PMF cannot be determined from marginal PMF.

Example

- A two-dimensional sequence is given by

$$f_{X,Y}[i,j] = c(1-p_1)^i(1-p_2)^j \qquad i = 1,2,\dots; j = 1,2,\dots$$

where $0 < p_1 < 1, 0 < p_2 < 1$ and c is a constant. Find c.

Example

- A two-dimensional sequence is given by

$$f_{X,Y}[i,j] = c(1-p_1)^i(1-p_2)^j \qquad i = 1,2,\ldots; j = 1,2,\ldots$$

where $0 < p_1 < 1, 0 < p_2 < 1$ and c is a constant. Find c.

- From the property of PMF we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c(1-p_1)^i(1-p_2)^j = 1$$

$$c\sum_{i=1}^{\infty}(1-p_1)^i\sum_{j=1}^{\infty}(1-p_2)^j = 1$$

$$c\frac{1-p_1}{1-(1-p_1)}.\frac{1-p_2}{1-(1-p_2)} = 1$$

$$c = \frac{p_1p_2}{(1-p_1)(1-p_2)}$$

Home work

- A joint PMF is given as

$$f_{X,Y}[i,j] = \left(\frac{1}{2}\right)^{i+j} \qquad i = 1, 2, \ldots; j = 1, 2, \ldots$$

If $A = \{(i,j) : 1 \leq i \leq 3; j \geq 2\}$ Find $P[A]$

Independence

Independence of Multiple Random Variables

- Outcome of coin toss, X. Outcome of die toss, Y
  - Y doesnt depend upon X: Independent variables.
  - Similarly, probabilites are independent. For Discrete RV

$$P[X \in A, Y \in B] = P[X \in A]P[Y \in B]$$

  - If $A = x_i$ and $B = y_i$

$$\begin{aligned} P[X \in A, Y \in B] &= P[X = x_i, Y = y_j] \\ &= P[X = x_i]P[Y = y_j] \\ f_{XY}[x_i, y_j] &= f_X[x_i]f_Y[y_j] \end{aligned}$$

For Continuous RV

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

$$F_{XY}(x, y) = F_X(x) F_Y(y)$$

- For independent random variables, joint PMF/PDF factors to product of Marginal PMF's/PDF's.
- If the joint PMF factors, the random variables are independent.

Example

- Determine the joint CDF if X and Y are independent with

$$
\begin{aligned}
f_X(x) &= \frac{1}{2} \qquad && 0 < x < 2 \\
&= 0 && \text{otherwise} \\
f_Y(y) &= \frac{1}{4} && 0 < y < 4 \\
&= 0 && \text{otherwise}
\end{aligned}
$$

- Determine the joint PDF first.

Example

- Determine the joint CDF if X and Y are independent with

$$
\begin{aligned}
f_X(x) &= \frac{1}{2} && 0 < x < 2 \\
&= 0 && \text{otherwise} \\
f_Y(y) &= \frac{1}{4} && 0 < y < 4 \\
&= 0 && \text{otherwise}
\end{aligned}
$$

- Determine the joint PDF first.
- Since X and Y are independent...

$$
\begin{aligned}
f_{XY}(x, y) &= f_X(x) f_Y(y) \\
&= \frac{1}{8} \qquad 0 < x < 2, 0 < y < 4.
\end{aligned}
$$

Example

- Joint CDF will be given as

$$F_{XY}(x,y) = \int_0^x \int_0^y \frac{1}{8} dydx = \frac{1}{8}xy$$

- Summarise your results.

$$F_{XY}(x,y) = \frac{1}{8}xy \qquad 0 \le x < 2, 0 \le y < 4.$$

$$F_{XY}(x,4) = \frac{1}{2}x \qquad 0 \le x < 2, y > 4.$$

$$F_{XY}(2,y) = \frac{1}{4}y \qquad x > 2, 0 \le y < 4.$$

$$F_{XY}(x,y) = 0 \qquad x \le 0; y \le 0$$

$$F_{XY}(x,y) = 1 \qquad x \ge 2; y \ge 4$$

Conditional distributions

Conditional Distributions

- Recall, conditional probability was given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A|B)P(B) = P(A \cap B)$$

- For random variables we have conditional PDF/PMF

$$P(Y = y|X = x)P(X = x) = P(X = x, Y = y)$$

$$P(Y = y|X = x)f_X(x) = f_{XY}(x, y)$$

$$f_{Y|X}(y|x)f_X(x) = f_{XY}(x, y)$$

$$f_{Y|X}[y_j|x_i] = \frac{f_{XY}[x_i, y_j]}{f_X[x_i]}$$

Properties: Conditional Distributions

- Joint PMF/PDF can yield conditional PMF/PDF. Now,

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

- Using joint PDF we can obtain marginal PDF and therefore,

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{\int_{-\infty}^{\infty} f_{XY}(x, y)dy}$$

- Conditional PDF can be obtained using the joint PDF.

Properties: Conditional Distributions

- Conditional PMF's/PDF's are related (using Bayes rule)

$$P(B|A)P(A) = P(AB)$$

$$f_{Y|X}(y|x)f_X(x) = f_{XY}(x,y)$$

$$P(A|B)P(B) = P(AB)$$

$$f_{X|Y}(x|y)f_Y(y) = f_{XY}(x,y)$$

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)}$$

$$f_{Y|X}[y_j|x_i] = \frac{f_{X|Y}[x_i|y_j]f_Y[y_j]}{f_X[x_i]}$$

Properties: Conditional Distributions

- Conditional PMF's/PDF's are expressible using Bayes Rule

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{XY}(x,y)dy}$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{X|Y}(x|y)f_Y(y)dy}$$

Example

- The joint PDF of 2 noise voltages is known to be

$$f_{XY}(x, y) = \frac{1}{2\pi} \exp -(y^2 - xy + x^2/2) \qquad -\infty < x < \infty, -\infty < y < \infty$$

Find the marginal PDF of x and the conditional PDF $f_{Y|X}(y|x)$.

Example

- The joint PDF of 2 noise voltages is known to be

$$f_{XY}(x,y) = \frac{1}{2\pi}\exp -(y^2 - xy + x^2/2) \qquad -\infty < x < \infty, -\infty < y < \infty$$

Find the marginal PDF of x and the conditional PDF $f_{Y|X}(y|x)$.

- The marginal PDF of x is given as

$$\begin{aligned} f_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp -\left(y^2 - xy + \frac{x^2}{2}\right)dy \\ &= \frac{1}{\pi}\int_{0}^{\infty} \exp -\left(y^2 - xy + \frac{x^2}{2}\right)dy \\ &= \frac{1}{2\sqrt{\pi}}\exp\left(-\frac{x^2}{4}\right)\left[\frac{2}{\sqrt{\pi}}\int_0^{\infty} \exp -\left(\left(y - \frac{x}{2}\right)^2\right)dy\right] \\ &= \frac{1}{2\sqrt{\pi}}\exp\left(-\frac{x^2}{4}\right)\left[\frac{2}{\sqrt{\pi}}\int_0^{\infty} \exp(-\lambda^2)d\lambda\right] \end{aligned}$$

Example

$$= \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{x^2}{4}\right) erfc(x)$$
$$= \frac{1}{2\sqrt{\pi}} \exp(-\frac{x^2}{4}).$$

- The conditional PDF is given by

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_x(X)}$$

- Therefore,

$$f_{Y|X}(y|x) = \frac{\frac{1}{2\pi} \exp -\left(y^2 - xy + \frac{x^2}{2}\right)}{\frac{1}{2\sqrt{\pi}} \exp\left(-\frac{x^2}{4}\right)}$$
$$= \frac{1}{\sqrt{\pi}} \exp -\left(y^2 - xy + \frac{x^2}{4}\right)$$

Joint Functions

Joint Moments

- For a cont. RV, $k - l^{th}$ joint moments are given by

$$E_{X,Y}[X^k Y^l] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^k y^l f_{XY}(x,y)dxdy$$

- Similarly for a disc. RV, $k - l^{th}$ joint moments are given by

$$E_{X,Y}[X^k Y^l] = \sum_i \sum_j x_i^k y_j^l f_{XY}[x_i, y_j]$$

- This can be generalised to n random variables.

Joint Characteristic functions

- Recall that, a characteristic function of a single r.v was defined:

$$\begin{aligned}\phi(\omega) &= E[\exp(j\omega x)] \\ &= \int_{-\infty}^{\infty} \exp(j\omega x) f_X(x) dx\end{aligned}$$

- For multiple random variables the joint characteristic function is written as

$$\begin{aligned}\phi_{X,Y}(\omega_x, \omega_y) &= E_{X,Y}[\exp[j(\omega_x x + \omega_y y)]] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) \exp[j(\omega_x x + \omega_y y)] dxdy\end{aligned}$$

- Again, this can be generalised to n random variables.

Moment generating functions

- Recall that, characteristic functions were used to obtain n^{th} moments

$$\frac{1}{j^n}\frac{d^n\phi(\omega)}{d\omega^n}\bigg|_{\omega=0} = E[X^n]$$

- $k-l^{th}$ moment is obtained by differentiating

$$E_{XY}[X^kY^l] = \frac{1}{j^{k+l}}\frac{\partial^{k+l}\phi_{X,Y}(\omega_x,\omega_y)}{\partial\omega_x^k\partial\omega_y^l}\bigg|_{\omega_x=\omega_y=0}$$

N-dimensional random variables

N-dimensional random variables.

- An N - dimensional vector denoted by $\mathbf{X} = [X_1, X_2, \ldots X_n]^T$
- Mapping from the original sample space of the experiment to a numerical sample space $\mathcal{S}_{X_1, X_2, \ldots, X_N} = \mathbb{R}^N$ such that

$$\mathbf{X}(s) = \begin{pmatrix} X_1(s) \\ X_2(s) \\ . \\ . \\ X_n(s) \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$$

Joint Distributions for n-dim RV

- Joint CDF:

$$F_{\mathbf{X}}(\mathbf{X}) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n)$$

- Joint PMF:

$$f_{\mathbf{X}}[\mathbf{X}] = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

- Joint PDF:

$$f_{\mathbf{X}}(\mathbf{X}) = \frac{\partial^n F_X(\mathbf{X})}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

Independence of n-dim RV

- Mutual independence holds iff

$$F_{\mathbf{X}}(\mathbf{X}) = F_{X_1}(x_1)F_{X_2}(x_2)\dots F_{X_n}(x_n)$$

$$f_{\mathbf{X}}(\mathbf{X}) = f_{X_1}(x_1)f_{X_2}(x_2)\dots f_{X_n}(x_n)$$

Affine transformation

- Recall for single random variable $y = ax + b$, we had

$$f_y(y) = \frac{1}{|a|} f_x\left(\frac{y-b}{a}\right) \tag{1}$$

- Similarly, if $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$, $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$ is invertible matrix. Then PDF of $\mathbf{y}$ is given by

$$f_y(\mathbf{y}) = \frac{f_x(\mathbf{A}^{-1}(\mathbf{y} - \mathbf{b}))}{|\mathbf{A}|} \tag{2}$$

Expectation

- Expectation extended to n-dimensional random variables.

$$E[\mathbf{X}] = \begin{pmatrix} E[X_1] \\ E[X_2] \\ . \\ . \\ E[X_n] \end{pmatrix}, \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ . \\ . \\ \mu_n \end{pmatrix}$$

Expectation

- For a function $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the expectation operator of the vector r.v. X generalises to

$$E[g(\mathbf{X})] = \int_{\mathbb{R}^n} g(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

- For the affine transformation we have

$$E[\mathbf{AX} + \mathbf{b}] = \mathbf{A}E[\mathbf{X}] + \mathbf{b}$$

- For independent random variables

$$E[X_1, X_2, \ldots X_n] = E[X_1]E[X_2] \ldots E[X_n]$$

Covariance

$$var(X) = E[(X - E[X])^2]$$

$$var(X+Y) = E_{X,Y}[\{(X - E_X[X]) + (Y - E_Y[Y])\}^2]$$

$$var(X+Y) = E_{X,Y}[\{(X - E_X[X])^2 + (Y - E_Y[Y])^2 + 2(X - E_X[X])(Y - E_Y[Y])\}]$$

$$= E_X[\{(X - E_X[X])^2] + E_Y[(Y - E_Y[Y])^2] + 2E_{X,Y}[(X - E_X[X])(Y - E_Y[Y])\}]$$

$$var(X+Y) = var(X) + var(Y) + 2cov(X,Y)$$

- Thats why variance is not a linear operation

Covariance Matrix

- Matrix of covariances of random variables $X_1, X_2, \dots X_n$ will be.

$$\Sigma = \begin{pmatrix} E[(X_1-\mu_1)^2] & \dots & \dots & E[(X_1-\mu_1)]E[(X_n-\mu_n)] \\ & \ddots & & \\ E[(X_n-\mu_n)]E[(X_1-\mu_1)] & \dots & \dots & E[(X_n-\mu_n)^2] \end{pmatrix}$$

- Diagonal elements give variances, off-diagonal elements are covariances.

Correlation coefficient

- How far the random variables are correlated.

$$\rho_{X,Y} = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}}$$

- If $\rho_{X,Y} = 0$, X and Y are not correlated
- If $cov(X,Y)$ is zero, $\rho_{X,Y} = 0$
- Independent variables are always uncorrelated.

Linear transformation

Gaussian PDF

- A std. bivariate Gaussian PDF is defined as

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{\{x^2 - 2\rho xy + y^2\}}{2(1-\rho^2)}\right)$$

- ρ is the correlation coefficient between X and Y.
- Now, assume a linear transformation where two continuous RV's, X and Y, map onto 2 new continous RV's W and Z.

$$\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{pmatrix} \sigma_w & 0 \\ 0 & \sigma_z \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\begin{pmatrix} W \\ Z \end{pmatrix} = \mathbf{G} \begin{pmatrix} X \\ Y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{G}^{-1} \begin{pmatrix} W \\ Z \end{pmatrix}$$

Linear transformation of Gaussian PDF

$$\mathbf{G} = \begin{pmatrix} \sigma_w & 0 \\ 0 & \sigma_z \end{pmatrix} \quad \text{and} \quad \mathbf{G}^{-1} = \begin{pmatrix} 1/\sigma_w & 0 \\ 0 & 1/\sigma_z \end{pmatrix}$$

- Now,

$$\mathbf{G}^{-1} \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} w/\sigma_w \\ z/\sigma_z \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

- We now want to find $f_{WZ}(w, z)$.

$$f_{WZ}(w, z) = f_{XY}\left(\mathbf{G}^{-1} \begin{pmatrix} w \\ z \end{pmatrix} \right) |\det G^{-1}|$$

- $\det(\mathbf{G}^{-1}) = \frac{1}{\sigma_w \sigma_z}$

Linear transformation of Gaussian PDF

- From,

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(- \frac{\{x^2 - 2\rho xy + y^2\}}{2(1-\rho^2)} \right)$$

- we have

$$f_{WZ}(w,z) =$$

$$\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(- \frac{\{(\frac{w}{\sigma_w})^2 - 2\rho(\frac{w}{\sigma_w})(\frac{z}{\sigma_z}) + (\frac{z}{\sigma_z})^2\}}{2(1-\rho^2)} \right) \frac{1}{\sigma_w \sigma_z}$$

Linear transformation of Gaussian PDF

- But this PDF can also be written in vector matrix form as

$$f_{WZ}(w,z) = \frac{1}{2\pi \det^{1/2}(\boldsymbol{\Sigma})} \exp\left(-\frac{1}{2} \begin{pmatrix} w \\ z \end{pmatrix}^T \boldsymbol{\Sigma}^{-1} \begin{pmatrix} w \\ z \end{pmatrix} \right)$$

- $\boldsymbol{\Sigma}$ is the covariance matrix given by

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_w^2 & \rho\sigma_w\sigma_z \\ \rho\sigma_w\sigma_z & \sigma_z^2 \end{pmatrix}$$

Example

- If a joint PDF is given as

$$f_{XY}(x, y) = \left(\frac{1}{4}\right)^2 \exp\left\{\frac{1}{2}(|x|+|y|)\right\} \quad -\infty < x < \infty; -\infty < y < \infty$$

- Find the joint PDF $f_{WZ}(w, z)$ when

$$\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

- We saw that

$$f_{WZ}(w, z) = f_{XY}\left(\mathbf{G}^{-1} \begin{pmatrix} w \\ z \end{pmatrix}\right) |\det G^{-1}|$$

Example

- Here

$$\mathbf{G} = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

- Therefore

$$\mathbf{G}^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & -1 \end{pmatrix}$$

$$\det(\mathbf{G}^{-1}) = -\frac{1}{2}$$

Example

- We have

$$f_{WZ}(w,z) = f_{XY}\left(\mathbf{G}^{-1}\begin{pmatrix} w \\ z \end{pmatrix}\right)|\det G^{-1}|$$

- Hence

$$f_{WZ}(w,z) = -\frac{1}{2} f_{XY}\left(\begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix}\right)$$

$$f_{WZ}(w,z) = -\frac{1}{2} f_{XY}\left(\begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix}\right)$$

$$f_{WZ}(w,z) = -\frac{1}{2} f_{XY}\left(-\frac{1}{2}w + z, w - z\right)$$

Example

- Now the question was

$$f_{XY}(x,y) = \left(\frac{1}{4}\right)^2 \exp\left\{\frac{1}{2}(|x|+|y|)\right\} \quad -\infty < x < \infty; -\infty < y < \infty$$

- We obtained

$$f_{WZ}(w,z) = -\frac{1}{2} f_{XY}\left(-\frac{1}{2}w + z, w - z\right)$$

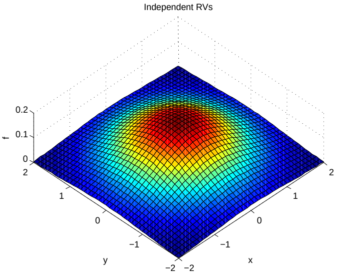
$$f_{WZ}(w,z) = \left(\frac{1}{4}\right)^2 \left(-\frac{1}{2}\right) \exp\left\{\frac{1}{2}(|-\frac{1}{2}w+z| + |w-z|)\right\}$$
$$= -\frac{1}{32}\exp\{\frac{1}{2}(|-\frac{1}{2}w+z| + |w-z|)\} \quad -\infty < x, y < \infty$$

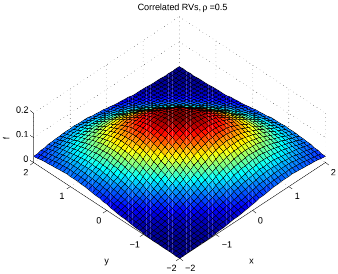
Multivariate Gaussian RV

- Joint PDF is given by

$$f_{\mathbf{X}}(\mathbf{X}) = \frac{1}{(2\pi)^{N/2}|\Sigma|^{1/2}} \exp\left\{\frac{-(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{2}\right\}$$

- $\boldsymbol{\mu}$ is the mean vector, $\mu_i = E[X_i]$
- $\boldsymbol{\Sigma}$ is the covariance matrix.





Independent and Identically distributed RV

- When $X_1, X_2, \ldots, X_n$ are independent RV's with same marginal PDF they are IID.
- Therefore, $E_{X_i}[X_i] = \mu$ and $var(X_i) = \sigma^2$ for all i.
- Averaged RV is given by

$$\overline{X}_N = \frac{1}{N}\sum_{i=1}^{N} X_i$$
$$= \frac{S_N}{N}$$

Expectation of IID RV

- Mean of $\overline{X}_N$

$$
\begin{aligned}
E_{\mathbf{X}}[\overline{X}_N] &= \frac{1}{N}\sum_{i=1}^{N} E_{\mathbf{X}}[X_i] \\
&= \frac{1}{N}\sum_{i=1}^{N} E_{X_i}[X_i] \\
&= \frac{1}{N} N\mu \\
&= \mu
\end{aligned}
$$

- Sample average converges towards the expected value. Law of Large Numbers.

IID Variance

- Variance is given by

$$
\begin{aligned}
Var(\overline{X}_N) &= E[(\overline{X}_N - E[\overline{X}_N])^2] \\
&= var\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right) \\
&= \frac{1}{N^2} var\left(\sum_{i=1}^{N} X_i\right) \\
&= \frac{\sigma^2}{N}
\end{aligned}
$$

- As $N \to \infty$, $var(\overline{X}) \to 0$, PDF will become more concentrated about μ.

Law of Large Numbers

- Therefore, if $X_1, X_2, \ldots, X_n$ are IID's with $E[X] = \mu$ and $var(X) = \sigma^2$ then $\lim_{N\to\infty} \overline{X_N} = E_X[X]$

$$
\begin{aligned}
P(|\overline{X}_N - E_X[X]| > \gamma) &\leq \frac{var(\overline{X}_N)}{\gamma^2} \\
&\leq \frac{\sigma^2}{N\gamma^2} \\
\lim_{N\to\infty} P(|\overline{X}_N - E_X[X]| > \gamma) &\leq \lim_{N\to\infty} \frac{\sigma^2}{N\gamma^2} \\
&= 0
\end{aligned}
$$

- Weak Law of large nos.
- Can be improved to $P(\lim_{N\to\infty} \overline{X_N} = \mu_x) = 1$. Strong law of large nos.

Central Limit Theorem

- Suppose $\mathbf{X}$ is a sequence of IID random variables with common mean (μ_x) and common variance (σ_x^2). Let these i.i.d random variables be normalised as

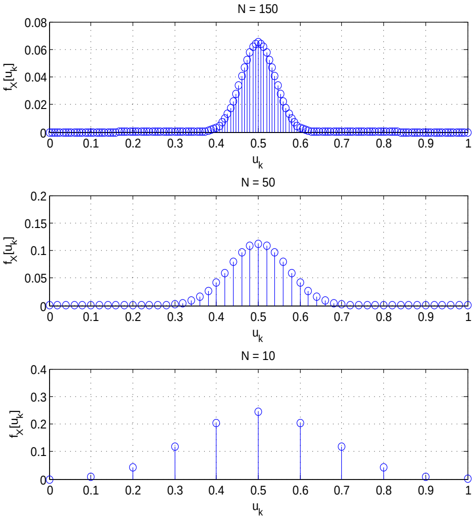
$$Y_i = \frac{1}{\sigma_x}(X_i - \mu_x) \;\text{ so that }\; E[Y_i] = 0 \;\text{ and }\; var[Y_i] = 1$$

- If we define the random variable

$$V_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i$$

- The central limit theorem states that as $N \to \infty$ the pdf of V_N approaches a normal distribution $\mathcal{N}(0, 1)$

Example: Central Limit Theorem



Central Limit Theorem (also stated as)

- Suppose $\mathbf{X}$ is a sequence of IID random variables with common mean and common variance. If $S_N = \sum_{i=1}^{N} X_i$ then as $n \to \infty$ we have

$$\frac{S_N - NE_X[X]}{\sqrt{Nvar(X)}} \to \mathcal{N}(0,1)$$

$$\lim_{N\to\infty} P\left(\frac{S_N - NE_X[X]}{\sqrt{Nvar(X)}} \le x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2}\right) du$$

- Note, Gaussian Distribution

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \quad -\infty < x < \infty$$