ENGG7302:

Advanced Computational Techniques in Engineering

Lecture 4-5: Stochastic Process

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April 11, 2011

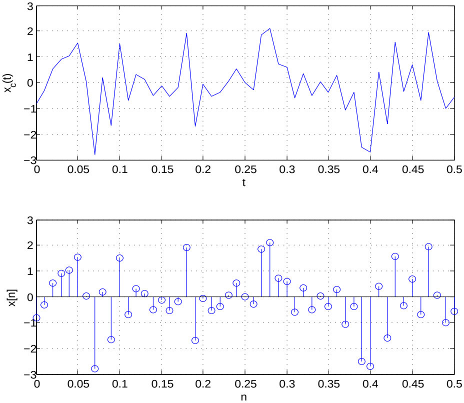
Overview of this lecture

- Signal
- Operations
- Stationary Processes
- Systems
- LTI systems
- Power Spectral Density
- Cross Spectral Density
- Filtering Stochastic Processes

Signals

- Function of one or more variables which conveys information about a phenomenon e.g. time, space, frequency.
- Continuous time Signals
    - Defined for all real valued instants of time, $x(t)$
- Discrete-time Signals
    - Defined for discrete instants of time.
    - Mathematically represented as sequence of numbers, $\mathbf{x}$.
    - n^{th} number in the sequence is represented as $x[n]$
    - Obtained by periodic sampling of an analog signal.

Continuous and Discrete signals



- ‘plot(x,y)’ and ‘stem(x,y)’ functions in Matlab.

Systems

- Deterministic
  - No randomness involved
  - Output of system can be predicted.
- Stochastic
  - Output of such a system is totally random, unpredictable
  - tossing of a coin, communication systems.

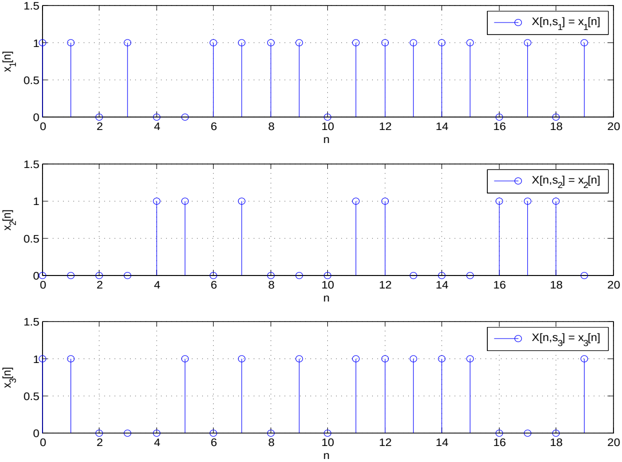
Random process example

- Mapping the outcomes from the original experimental sample space.

$$\begin{aligned} \mathcal{S} &= \{(H, H, T, \dots), (H, T, H, \dots), (T, T, H, \dots), \dots\} \\ \mathcal{S}_X &= \{(1, 1, 0, \dots), (1, 0, 1, \dots), (0, 0, 1, \dots), \dots\} \\ &= \{\mathbf{X}[0], \mathbf{X}[1], \mathbf{X}[2], \dots\} \end{aligned}$$

- with each outcome of the random process denoted as $\mathbf{X}[0] = (x[0], x[1], \dots)$
- The random process is a mapping from $\mathcal{S}$ which is a set of infinite and sequential experimental outcomes to $\mathcal{S}_X$ which is an infinte sequence of realisations.

Random process

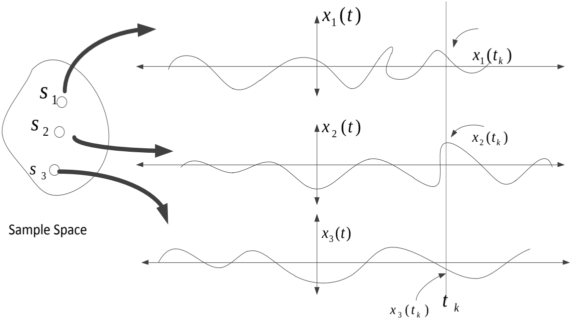


Random process

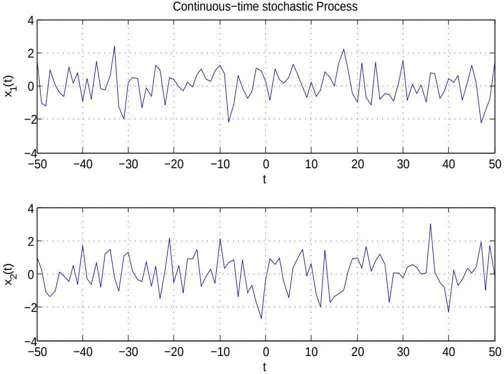
- A sample space or ensemble composed of functions of time is called Random or stochastic process
- Suppose $s_1, s_2, \ldots, s_n$ form the sample points of a sample space $\mathcal{S}$

$$X(t, s), -T \leq t \leq T$$

$$x_j(t) = X(t, s_j)$$

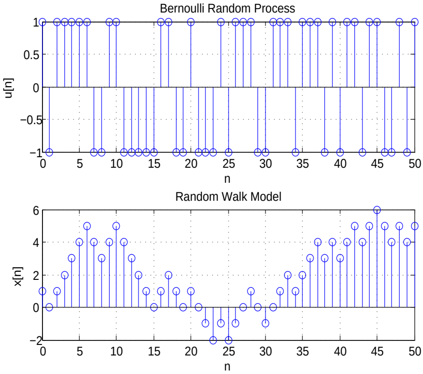


Continuous time stochastic process



Random process

- Using Bernoulli random process, we simulate a random Walk Model $X_n = \sum_{i=1}^{N} U_i$



Random Processes

- Vehicle travel (force acting on the wheel of a car).
- Plane turbulence.
- Vibrations.
- Wind shear at different times.
- Signal interferences.

Statistical properties

Statistical properties: Random process

- Statistical properties of a stochastic process determined by $n^{th}-$order distributions

$$F_{X(t_1),\ldots,X(t_n)} = P(X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n)$$

- Random Number: Outcome of an experiment is mapped into a number
- Random Process:Outcome of an experiment is mapped into a waveform which is a function of time.

Operations

- Mean

$$\begin{aligned}\mu_x(t) &= E[X(t)] \\ &= \int_{-\infty}^{\infty} x f_{X(t)}(x) dx\end{aligned}$$

- Autocorrelation

$$\begin{aligned}R_x(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1)X(t_2)}(x_1 x_2) dx_1 dx_2\end{aligned}$$

Operations

- Autocovariance

$$C_x(t_1, t_2) = E[\{X(t_1) - \mu_x(t_1)\}\{X(t_2) - \mu_x(t_2)\}]$$
$$= R_X(t_1, t_2) - \mu_x(t_1)\mu_x(t_2)$$

- Crosscorrelation

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X(t_1)Y(t_2)}(xy)dxdy$$

Operations

- Cross covariance

$$C_{XY}(t_1, t_2) = E[\{X(t_1) - \mu_x(t_1)\}\{Y(t_2) - \mu_y(t_2)\}]$$
$$= R_{XY}(t_1, t_2) - \mu_x(t_1)\mu_y(t_2)$$

- Complex Stochastic Process

$$Z(t) = X(t) + jY(t)$$
$$R_Z(t_1, t_2) = E[Z(t_1)Z^*(t_2)]$$

Types of Random Processes

Strict Sense Stationary Process

- A random process $X(t)$. At times $t_1, t_2, \ldots, t_n$, we observe random variables $X(t_1), X(t_2), \ldots, X(t_n)$.
- The joint p.d.f is $f_{X(t_1),X(t_2),\ldots,X(t_n)} x_1 x_2 \ldots x_n$.
- A random process is strict sense stationary if

$$f_{X(t_1+\tau),X(t_2+\tau),\ldots,X(t_n+\tau)}(x_1 x_2 \ldots x_n) = f_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1 x_2 \ldots x_n)$$

- The joint p.d.f of random variables obtained by observing the random process is invariant w.r.t the location of the origin $t = 0$

IID random process

- So is IID random process stationary?
- Yes, Marginal PDF is the same for each RV. Therefore

$$f_{X(t_1+\tau),X(t_2+\tau),\ldots,X(t_n+\tau)}(x_1x_2\ldots x_n) = f_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1x_2\ldots x_n)$$

- Any process whose mean or variances change with time is NOT stationary.

Wide sense Stationary Process

- Using joint p.d.f, first order distribution function, i.e. for n = 1 will be

$$f_{X(t_1+\tau)}(x) = f_{X(t_1)}(x)$$

- If $t_1 = 0$, then

$$f_{X(\tau)} = f_{X(0)}(x) \quad \forall \tau$$

- Since the PDF does not depend on a particular time, so should not mean/expectation. Therefore

$$E[X(t)] = \int_{-\infty}^{\infty} x f_X(x) dx = \mu_x = \text{constant}$$

- Mean of a wide-sense stationary process is constant

Wide sense stationary process

- Now, for $n = 2$ we have:

$$f_{X(t_1+\tau)X(t_2+\tau)}(x_1x_2) = f_{X(t_2)X(t_1)}(x_1x_2) \qquad \forall t_1, t_2$$

- If $\tau = -t_1$ we have,

$$f_{X(t_1-t_1)X(t_2-t_1)}(x_1x_2) = f_{X(t_2)X(t_1)}(x_1x_2) \qquad \forall t_1, t_2$$

- This leads to

$$E[X(t_1)X(t_2)] = E[X(0)X(t_2-t_1)]$$

$$R_X(t_1, t_2) = R_X(t_2 - t_1)$$

- Such a process is also called weakly stationary.

Covariance

- Autocovariance of a WSS process is

$$C_X(t_1, t_2) = E[\{X(t_1) - \mu_x\}\{X(t_2) - \mu_x\}]$$
$$= R_x(t_2 - t_1) - \mu_X^2$$

- A second order stationary process can be wide-sense stationary but the converse is not true.

Properties of correlation for WSS process

- An Autocorrelation function is defined as

$$R_X(\tau) = E[X(t+\tau)X(t)] \forall t$$

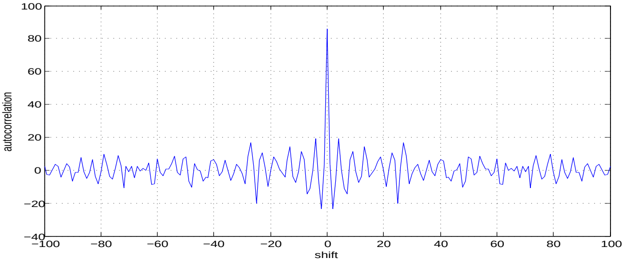
- Mean square value of the process can be obtained by $\tau = 0$.

$$R_X(0) = E[X^2(t)] \quad \text{this is second moment}$$

Properties

- Autocorrelation is an even function

$$R_X(\tau) = R_X^*(-\tau)$$

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>> x = randn(1,101);
>> [a, shift] = xcorr(x);
>> plot(shifts,a);
```

Properties of correlation

- Autocorrelation function $R_X(\tau)$ has the maximum magnitude at $\tau = 0$

$$E[(X(t+\tau) + X(t))^2] \geq 0$$
$$E[X^2(t+\tau)] + E[X^2(t)] + 2E[X(t+\tau)X(t)] \geq 0$$
$$2R_X(0) + 2R_X(\tau) \geq 0$$
$$R_X(0) \geq |R_X(\tau)|$$

Joint Stationary Process

- A pair of stochastic processes X(t) and Y(t) are joint stationary if any joint distribution of X(t) and Y(t) is insensitive to a time shift.
- They are jointly wide sense stationary (WSS) if each is WSS and if the cross-correlation depends only on time difference.

Joint stationary

- For jointly WSS processes,

$$R_{XY}(\tau) = R^*_{YX}(-\tau)$$

- If for all τ, $R_{XY}(\tau) = 0$ we say X(t) and Y(t) are orthogonal (Eg. 1.4, Comm. systems Simon Haykin).
- If for all τ, $C_{XY}(\tau) = 0$ we say X(t) and Y(t) are uncorrelated.

Example

- Consider a pair of quadrature modulated process that are related to stationary process as

$$X_1(t) = X(t)\cos(2\pi f_c t + \theta)$$
$$X_2(t) = X(t)\sin(2\pi f_c t + \theta)$$

where f_c is a carrier frequency and θ is a random variable uniformly distributed over the interval $[0, 2\pi]$ and independent of $X(t)$. Obtain the cross correlation.

White noise

- White noise is a WSS random process with zero mean, identical variance and uncorrelated samples.
- Autocorrelation of a white noise sequence

$$
\begin{aligned}
R_x[k] &= E[X[n]X[n+k]] \\
&= E[X[n]]E[X[n+k]] \\
&= 0 \qquad\qquad k \neq 0 \;\; \text{uncorr. } 0 \text{ mean} \\
&= E[X^2[n]] \qquad k = 0 \\
&= \sigma^2 \\
\text{Hence, } \; R_x[k] &= \sigma^2 \delta[k]
\end{aligned}
$$

- A stochastic process is white noise when $C_x(t_1, t_2) = 0$ when $t_1 \neq t_2$

Some Examples

- (1) Obtain $R_Y(\tau)$ if

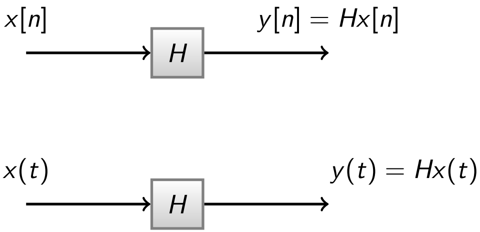
$$Y(t) = X(t+a) - X(t-a)$$

- (2) Show that

$$R_X(0) + R_Y(0) \geq 2|R_{XY}(\tau)|$$

System

- A system is defined as interconnection of operations that transform input signal to output signal



Types of Systems

- Discrete time System

$$\begin{aligned} y[n] &= h[n] * x[n] \\ &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \end{aligned}$$

- Continuous time System

$$\begin{aligned} y[n] &= h(t) * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau) \end{aligned}$$

Example: Convolution

- Discrete Convolution
  - Convolve $x[n] = [1, 2, 1, -1]$ and $h[n] = [1, 2, 3, 1]$
- Continuous convolution

$$x(t) = 4\delta(t-1)$$

$$h(t) = 2\delta(t) + 2\delta(t-4)$$

- Convolve

$$x(t) = 4\Pi\left(\frac{t}{2} - \frac{1}{2}\right)$$

$$h(t) = 2\Pi\left(t - \frac{1}{2}\right)$$

Memory and Linearity

- Output depends not only on the current values of input but also on the past or future values

$y[n] = x[n] - x[n-1]$ Has memory

$y[n] = 3 - x[n]$ No memory

- Linear Systems
  - Have the properties of homogenity (scaling) and/or additivity.

$$y[n] = x[n] \pm x[n-1]$$

$$y[n] = 4x[n]$$

Causal and Stable

- Causal
  - Output depends on current and past values of input.

$y(t) = x(t) - x(t-1)$	Is causal
$y(t) = x(t) - x(t+1)$	Is acausal

- Stable
  - every bounded input produces a bounded output

$y[n] = nx[n]$	Not stable
$y(t) = x^2(t)$	Stable

Time variance/in variance

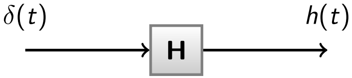
- If a time shift in the input causes a corresponding time shift in the output it is a time-invariant system

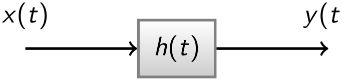
$$y(t) = x(t)$$
$$y(t-5) = x(t-5)$$

- Most communication systems are Linear Time Invariant.

LTI systems

- In time-domain, a linear system is defined in terms of its impulse response.
- i.e Response of a system (with zero initial conditions) to a unit impulse (or delta function) applied to the input.





$$y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Example: Ideal Delay

- Impulse response of an ideal delay

$$h[n] = \delta[n - n_d]$$

- Now the output of such a system is given as

$$\begin{aligned} y[n] &= h[n] * x[n] \\ &= \delta[n - n_d] * x[n] \\ &= x[n - n_d] \end{aligned}$$

Frequency Response

- Suppose $x(t) = \exp(jwt)$. Then

$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \exp\{jw(t-\tau)\}$$

$$= \exp(jwt) \int_{-\infty}^{\infty} h(\tau) \exp(-jw\tau)$$

$$\text{If} \quad H(w) = \int_{-\infty}^{\infty} h(\tau) \exp(-jw\tau)$$

$$y(t) = H(w) \exp(jwt)$$

Frequency Response

- An eigenfunction is a function for which the output of the operator is the same function, just scaled by some amount. In symbols,

$$\mathcal{H}f = \lambda f,$$

where f is the eigenfunction and λ is the eigenvalue, a constant.

- Complex exponential signal is an eigen function of an LTI system.

Frequency Response

- So for the eigen function $\exp(jwt)$, the eigen value is $H(w)$.
- The eigen value is called the frequency response of the system

$$\begin{aligned} H(w) &= H_R(w) + jH_I(w) \\ &= |H(w)| \exp(j\angle H(w)) \end{aligned}$$

Power Spectral Density

- Average power distributed as a function of frequency. Defined as

$$S_x(f) = \lim_{T\to\infty} \frac{1}{T} E\left[\left| \int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)\exp(-j2\pi ft)\right|^2\right]$$

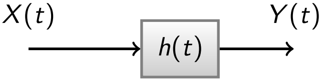
$$= \lim_{T\to\infty} \frac{1}{T} E[|X(f)|^2]$$

- Wiener Khintchine theorem states PSD is the FT of the autocorrelation sequence.

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau$$

Question

- For a Wide sense stationary process evaluate the mean square value of the output $Y(t)$.



$$
\begin{aligned}
R_Y(t,u) &= E[Y(t)Y(u)] \\
&= E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(u-\tau_2)d\tau_2\right]
\end{aligned}
$$

Power Spectral Density

$$R_Y(t,u) = \int_{-\infty}^{\infty} h(\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)d\tau_2 E[X(t-\tau_1)X(u-\tau_2)]$$

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)d\tau_2 R_X(\tau-\tau_1+\tau_2)$$

at $\quad \tau = 0, R_Y(0) = E[Y^2(t)]$

$$E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_2-\tau_1)d\tau_1 d\tau_2$$

Now, $\quad H(f) = \int h(\tau_1)\exp(-j2\pi f\tau_1)d\tau_1$ FFT?

Hence $\quad h(\tau_1) = \int H(f)\exp(j2\pi f\tau_1)df$ IFFT?

Power Spectral Density

$$E[Y^2(t)] = \int_{-\infty}^{\infty} H(f)df \int_{-\infty}^{\infty} h(\tau_2)d\tau_2 \int_{-\infty}^{\infty} R_X(\tau_2 - \tau_1)\exp(j2\pi f\tau_1)d\tau_1$$

Let $\tau = \tau_2 - \tau_1 \qquad \tau_1 = \tau_2 - \tau$

$$E[Y^2(t)] = \int_{-\infty}^{\infty} H(f)df \int_{-\infty}^{\infty} h(\tau_2)\exp(j2\pi f\tau_2)d\tau_2 \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau$$

$$E[Y^2(t)] = \int_{-\infty}^{\infty} H(f)H^*(f)df \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau$$

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau$$

$$E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)df$$

Power Spectral Density

- So we had

$$
\begin{aligned}
E[Y^2(t)] &= \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) df \\
S_Y(f) &= |H(f)|^2 S_X(f) \\
&= H(f)H^*(f)S_X(f) \\
R_Y(\tau) &= h(\tau) * h^*(-\tau) * R_X(\tau)
\end{aligned}
$$

Filtering Stochastic Process

- Consider a pair of LTI filters and process as shown.

- The cross correlation and cross spectral densities can be written as

$$R_{VZ}(\tau) = h_1(\tau) * h_2^*(-\tau) * R_{XY}(\tau)$$

$$S_{VZ}(f) = H_1(f)H_2^*(f)S_{XY}(f)$$

Properties of PSD

- $S_x(f)$ and $R_X(\tau)$ of a stationary process $X(t)$ form a Fourier transform pair (Wiener Khintchine theorem)

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau) d\tau \quad (1)$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) \exp(j2\pi f\tau) df \quad (2)$$

- PSD of a real valued random process is an even function of frequency

$$S_X(f) = S_X(-f)$$

Properties of PSD

- PSD is non-negative $S_X(f) \geq 0$.
- The zero frequency value of PSD equals total area under the graph of autocorrelation function. From (1), at $f = 0$

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau) d\tau$$

- Similarly, mean square value of a stationary process is the total area under the graph of PSD. From (2), at $\tau = 0$

$$E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f) df$$

Cross Spectral Density

- Measures the frequency relationship between 2 random processes.

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau) \exp(-j2\pi f \tau) d\tau$$

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} S_{XY}(f) \exp(j2\pi f \tau) df$$

Example

- Given 2 stationary processes $X(t)$ and $Y(t)$. If $Z = X(t) + Y(t)$. Find PSD of $Z(t)$.

$$
\begin{aligned}
R_Z(t,u) &= E[Z(t)Z(u)] \\
&= E[\{X(t)+Y(t)\}\{X(u)+Y(u)\}] \\
&= E[X(t)X(u) + X(t)Y(u) + X(u)Y(t) + Y(t)Y(u)] \\
&= R_X(t,u) + R_{XY}(t,u) + R_{YX}(t,u) + R_Y(t,u) \\
S_Z(f) &= S_X(f) + S_{XY}(f) + S_{YX}(f) + S_Y(f)
\end{aligned}
$$

Example

- If X and Y are uncorrelated

$$
\begin{aligned}
R_Z(t,u) &= E[Z(t)Z(u)] \\
&= E[\{X(t)+Y(t)\}\{X(u)+Y(u)\}] \\
&= E[X(t)X(u)+X(t)Y(u)+X(u)Y(t)+Y(t)Y(u)] \\
&= R_X(t,u)+0+0+R_Y(t,u) \\
S_Z(f) &= S_X(f)+S_Y(f)
\end{aligned}
$$

Example

- Consider a sinusoidal signal with random phase defined by

$$X(t) = A\cos(2\pi f_c t + \theta)$$

where A and f_c are constants and θ is a random variable uniformly distributed over the interval $[-\pi, \pi]$, that is,

$$f_\theta(\theta) = \frac{1}{2\pi} \quad -\pi \leq \theta \leq \pi$$
$$= 0 \quad \text{elsewhere}$$

Obtain $R_X(\tau)$ and the PSD.