ENGG7302:

Advanced Computational Techniques in Engineering

Lecture 5-6: Discrete-time Stochastic Processes

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Overview of this lecture

- Correlation and Covariance
- Z-Transform and DTFT
- Difference equations
- PSD
- Filtering
- Sampling

Correlation and covariance

- Most definitions of continuous-time stochastic process are applicable for discrete time
- Autocorrelation

$$R_X[n_1, n_2] = E[X[n_1]X^*[n_2]]$$

- Autocovariance

$$C_X[n_1, n_2] = E[\{X[n_1] - \mu_X[n_1]\}\{X[n_2] - \mu_X[n_2]\}^*]$$

- If $X[n]$ is strict sense stationary then joint distributions are invariant to time-shift.

Wide Sense stationary

- If $X[n]$ is wide sense stationary then

$$R_X[n_1, n_2] = R_X[n_1 - n_2]$$

$$\mu_X[n] = \mu_X = \text{constant}$$

- In a discrete time white noise, $X[n]$ are uncorrelated RV's.

Example

- Given the stationary random sequence, $x[n] = \{-1.6129, -1.2091, -0.4379, -2.0639, -0.6484\}$, estimate the mean, the variance, the autocorrelation with a lag 0.

$$\mu = \frac{-1.6129 - 1.2091 - 0.4379 - 2.0639 - 0.6484}{5}$$
$$= -1.1944.$$

Example

- Recall, SD is how much a RV deviates from the mean. Variance is SD^2. First subtract the mean from each sample of the sequence and we get $\{-0.4185, -0.0147, 0.7565, -0.8695, 0.5460\}$. Then

$$\sigma^2 = \frac{(-0.4185)^2 + (-0.0147)^2 + 0.7565^2 + (-0.8695)^2 + 0.5460^2}{5}$$
$$= 0.3604.$$

- Autocorrelation,

$$R_{XX}[0] = \frac{1.6129^2 + 1.2091^2 + 0.4379^2 + 2.0639^2 + 0.6484^2}{5}$$
$$= 1.787$$

Example: Autocovariance

- Autocorrelation lag of 1

$$R_{XX}[1] = \frac{-1.6129(-1.2091) - 1.2091(-0.4379) - 0.4379(-2.0639) - 2.0639(-0.6484)}{4}$$
$$= 1.1804$$

- Autocovariance with a lag of 0

$$C_{XX}[0] = 0.3604$$

- Autocovariance with a lag of 1

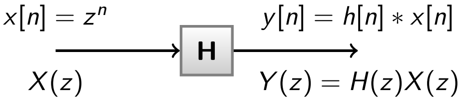
$$C_{XX}[1] = \frac{-0.4185(-0.0147) - 0.0147(0.7565) + 0.7565(-0.8695) - 0.8695(0.5460)}{4}$$
$$= -0.2844$$

Discrete-time signals and systems

- Recall that for continuous signals, $\exp(jwt)$ was an eigen function of an LTI system.
- For discrete it is $\exp(jwn)$.

Discrete-time signals and systems

- Recall that for continuous signals, $\exp(jwt)$ was an eigen function of an LTI system.
- For discrete it is $\exp(jwn)$.
- Eigen functions of Discrete LTI systems have the form $x[n] = z^n$.



Discrete-time Fourier Transform (DTFT) and Z

- Consider a signal $x[n]$. Its discrete time fourier transform is given by

$$X(w) = \sum_{n=-\infty}^{\infty} x[n] \exp(-jwn)$$

- But the Z transform of $x[n]$ is given by:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- Therefore, at $z = \exp(jw)$, the z-transform of $x[n]$ is the discrete time fourier transform.

Relationship between DTFT and DFT

- OTH, a discrete fourier transform is a sampled version of DTFT

$$X(w)\Big|_{k\Delta w} = X[k] = \sum_{n=-0}^{N-1} x[n] \exp\left(-\frac{jnk2\pi}{N}\right)$$

Z-transform

- Impulse response and the Transfer Function are related via the z-transform i.e.

$$h[n] \xrightarrow{Z} H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

- Using the convolution property we can write

$$Y(z) = H(z)X(z)$$
$$H(z) = \frac{Y(z)}{X(z)}$$

- $H(z)$ is the transfer function. What does it tell about the digital system?

Frequency Response

Frequency Response

- So we had $\exp(jwn)$ is the eigen function of the discrete system.
- Suppose $x[n] = \exp(jwn)$. Then

$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] \exp\{jw(n-k)\}$$

$$= \exp(jwn) \sum_{k=-\infty}^{\infty} h[k] \exp(-jwk)$$

$$\text{If} \quad H(\exp(jw)) = \sum_{k=-\infty}^{\infty} h[k] \exp(-jwk)$$

$$y[n] = H(\exp(jw)) \exp(jwn)$$

Frequency Response

- So for the eigen function $\exp(jwn)$, the eigen value is $H(\exp(jw))$.
- The eigen value is called the frequency response of the system

$$\begin{aligned} H(\exp(jw)) &= H_R(\exp(jw)) + jH_I(\exp(jw)) \\ &= |H(\exp(jw))| \exp(j\angle H(\exp(jw)) \end{aligned}$$

Power Spectral Density

- Recall, for continuous processes PSD was given by:

$$S_x(f) = \lim_{T\to\infty} \frac{1}{T} E\left[\left| \int_{-\frac{T}{2}}^{\frac{T}{2}} X(t) \exp(-j2\pi ft) \right|^2 \right]$$

$$= \lim_{T\to\infty} \frac{1}{T} E[|X(f)|^2]$$

- For discrete, PSD can be defined in terms of the DTFT.

$$S_X(\exp(jw)) = \lim_{N\to\infty} \frac{1}{2N+1} E[|X_N(\exp(jw))|^2]$$

$$X_N(\exp(jw)) = \sum_{n=-\infty}^{\infty} x[n] \exp(-jwn)$$

Power Spectral Density

- For a WSS process

$$R_X[m] \overset{DTFT}{\leftrightarrow} S_X(\exp(jw))$$

- Inverse DTFT can be used to obtain $R_X[m]$ from $S_X(\exp(jw))$

$$R_X[m] = \int_{-\pi}^{\pi} S_X(\exp(jw)) \exp(jwm)$$

Difference equations for LTI

LTI systems governed by difference equations

- The inputs and outputs of a LTI system can be written via a difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Taking Z-transforms

$$Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{k=0}^{M} b_k z^{-k}$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- H(z) becomes the transfer function.

FIR filters

- Finite Impulse response filters
- Represented by the difference equation.

$$y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_k x[n-k]$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

- b_k become the coefficients of the filters.
- Feedback stage absent
- Stable filters

Transfer function: FIR filters

- We had

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

- Taking Z transforms we have

$$Y(z) = X(z)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

Moving average process

- Impulse response of MA process is given by

$$h[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} \delta[n-k]$$

- Impulse response is finite. Now, the output is $h[n] * x[n]$. Therefore

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

IIR filters

- Infinte Impulse response filters
- Can be modelled using a difference equation given by

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

  a_k and b_k become the coefficients of the filters.
- Feedback stage is present

Transfer function: IIR filters

- We had

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

- Taking Z transforms we have

$$Y(z)\left(1 - \sum_{k=1}^{N} a_k z^{-k}\right) = X(z)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

Accumulator

- Impulse response of accumulator is given by

$$h[n] = \sum_{k=-\infty}^{n} \delta[k]$$

- Impulse response is infinite. Now, the output is $h[n] * x[n]$. Therefore

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Accumulator: Difference equation

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

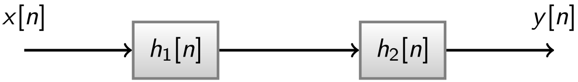
- This becomes an IIR filter of the form

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

Discrete time Processes

- If $N = 0$, and $M > 0$ becomes a Moving Average process.
- If $M = 0$, and $N > 0$ becomes an Auto regressive process.
- If $M > 0$ and $N > 0$, it becomes an autoregressive moving average process ARMA(M,N).

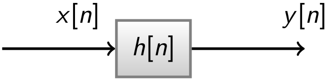
Cascaded systems



- The Impulse response of the overall system is given by

  $h[n] = h_1[n] * h_2[n]$

Filtering Stochastic processes



- When the input to a LTI system is WSS, the autocorrelations and PSDs of input X[n] and output Y[n] are related:

$$R_Y[m] = h[m] * h^*[-m] * R_X[m]$$

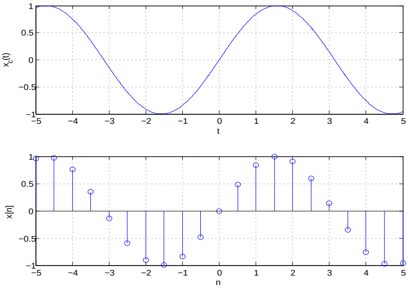
$$S_Y(\exp(jw)) = |H(\exp(jw))|^2 S_X(\exp(jw))$$

Sampling

Sampling

- An analogue signal $x_c(t)$ is sampled to produce a digital or discrete-time signal $x[n]$ such that

$$x[n] = x_c(nT_s)$$



Nyquist criteria

- To avoid alaising, a bandlimited signal of bandwidth w_B rad/s must be sampled at a frequency of $2w_B$ rad/s That is $w_s > 2w_B$.
- This is basis of operation for A/D and D/A convertors.

Sampling: Example

- For the signal $X(w) = u(w) - u(w - w_0)$ calculate the sampling period.
- For the signal $X(w) = u(w + w_0) - u(w - w_0)$ calculate the sampling period.