Question

Consider a sinusoidal signal with random phase defined by

$$X(t) = A \cos(2\pi f_c t + \theta) \quad \text{where}$$

A and f_c are constants and θ is a random variable uniformly distributed over the interval $(-\pi, \pi)$ that is.

$$f_\theta(\theta) = \frac{1}{2\pi} \qquad -\pi < \theta < \pi$$

$$= 0 \qquad \text{elsewhere.}$$

Obtain $R_x(\tau)$ and PSD

Solution :-

$$R_x(\tau) = E[x(t)\, x(t+\tau)]$$

$$= A^2 E[\cos(2\pi f_c t + \theta) \cos(2\pi f_c (t+\tau) + \theta)]$$

$$= A^2 E[\cos(2\pi f_c t + \theta) \cos(2\pi f_c t + 2\pi f_c \tau + \theta)]$$

Now $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$= \frac{A^2}{2} E[\cos(\underbrace{4\pi f_c t + 2\pi f_c \tau}_{A} + \underbrace{2\theta}_{B}) + \cos(2\pi f_c \tau)]$$

Again,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$= \frac{A^2}{2}\left\{E\left[\cos(4\pi f_c t + 2\pi f_c \tau)\cos 2\theta - \sin 2\theta \sin(4\pi f_c t + 2\pi f_c \tau)\right] + E\left[\cos(2\pi f_c \tau)\right]\right\}$$

$$= \frac{A^2}{2}\left\{\left[\cos(4\pi f_c t + 2\pi f_c \tau)\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos 2\theta \, d\theta - \sin(4\pi f_c t + 2\pi f_c \tau)\frac{1}{2\pi}\int \sin 2\theta \, d\theta\right] + \cos(2\pi f_c \tau)\right\}$$

Now,
$$\int_{-\pi}^{\pi}\cos 2\theta = \left.\frac{\sin 2\theta}{2}\right]_{-\pi}^{\pi}$$
$$= \underline{\underline{0}}$$

and,
$$\int_{-\pi}^{\pi}\sin 2\theta = -\left.\frac{\cos 2\theta}{2}\right|_{-\pi}^{\pi}$$
$$= -\frac{1}{2}\left(\cos 2\pi - \cos(-2\pi)\right)$$
$$= -\frac{1}{2}\left[\cos(2\pi) - \cos(2\pi)\right]$$
$$= 0 \quad \because \underline{\underline{\cos(-\theta) = \cos\theta}}$$

Hence
$$R_x(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

Now, PSD is F-T of R_x

$$\mathcal{F}\{R_x(\tau)\} = S_x(f)$$

Fourier Transform of $\cos(2\pi f_c \tau)$ is given as

$$= \int_{-\infty}^{\infty} \cos(2\pi f_c \tau) \exp(-j 2\pi f \tau)\, d\tau$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left(\exp(j 2\pi f_c \tau) + \exp(-j 2\pi f_c \tau)\right) \exp(-j 2\pi f \tau)\, d\tau$$

$$= \frac{1}{2} \int \exp(-j 2\pi (f - f_c) \tau) + \exp(-j 2\pi (f + f_c) \tau)\, d\tau$$

which is :-

$$= \frac{1}{2} \left[\delta(f - f_c) + \delta(f + f_c)\right]$$

Hence $S_x(f) = \mathcal{F}\{R_x(\tau)\}$

$$= \frac{A^2}{2} \left\{ \frac{1}{2}\, \delta(f - f_c) + \delta(f + f_c) \right\}$$

$$\boxed{S_x(f) = \frac{A^2}{4} \left[\delta(f - f_c) + \delta(f + f_c)\right]}$$