THE UNIVERSITY OF QUEENSLAND

VENUE:
SEAT NUMBER:
STUDENT NUMBER:

FINAL EXAMINATION

St Lucia Campus

First Semester, 2008

ENGG7302/ELEC4000 — Advanced Computational Techniques in Engineering

PERUSAL TIME	**10mins.**	**During perusal, write only on this exam paper.**
WRITING TIME	**180 minutes**	
EXAMINER	**A/Prof. Vaughan Clarkson and Dr. Marcus Gallagher**	

NO. OF PAGES (*include title page and attachments*) 7 Pages - Double-Sided

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Exam Type: **Closed Book - Specified materials permitted**

Permitted Materials:
Calculator - EPSA approved and labelled calculators only
Dictionary - Yes - Any unmarked paper dictionary is permitted
Other – No electronic aids are permitted (e.g. laptops, phone)

Answer: **In writing booklet**

Number of Questions: **8**

Weighting/Marks: **50% / 80 marks**

Special Instructions: **Students must comply with the General Award Rules 1A.7 and 1A.8 which outline the responsibilities of students during an examination.**

ANSWER ALL QUESTIONS

Question 1. (10 marks)

(a) Show that $\|\mathbf{Qx}\| = \|\mathbf{x}\|$ when $\mathbf{Q}$ is a unitary matrix. (2 marks)

(b) Show that, for a rank-1 matrix, the 2-norm and Frobenius norm are equal. (2 marks)

(c) Calculate the full SVD of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

(6 marks)

Question 2. (10 marks)

(a) Which of the following are orthogonal projection matrices and why?

$$\text{i)}\quad \mathbf{P} = \tfrac{1}{2}\begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \quad \text{ii)}\quad \mathbf{P} = \tfrac{1}{2}\begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}, \quad \text{iii)}\quad \mathbf{P} = \begin{pmatrix} 0 & j \\ 0 & 1 \end{pmatrix}.$$

(3 marks)

(b) A matrix $\mathbf{A}$ has the reduced QR decomposition $\mathbf{A} = \hat{\mathbf{Q}}\hat{\mathbf{R}}$ where

$$\hat{\mathbf{Q}} = \begin{pmatrix} 0.6 & 0 \\ 0 & 1 \\ 0.8 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{R}} = \begin{pmatrix} 23 & -8 \\ 0 & -15 \end{pmatrix}.$$

Compute a full QR decomposition. (2 marks)

(c) Consider a sphere of radius r with a centre at (u, v, w). A set of points (x_i, y_i, z_i), $i = 1, \ldots, N$, are observed, each slightly displaced from the surface of the sphere because of noise or errors in the measurement process. With

$$t = r^2 - u^2 - v^2 - w^2,$$

a procedure has been developed to estimate the parameters t, u, v, w through minimisation of the objective function

$$f(t, u, v, w) = \sum_{i=1}^{N} (x_i^2 - 2ux_i + y_i^2 - 2vy_i + z_i^2 - 2wz_i - t)^2.$$

i) Show how this function can be minimised using a pseudo-inverse. (3 marks)

ii) Given vectors `x`, `y`, `z`, write a few lines of MATLAB code to estimate the parameters. (2 marks)

Question 3. (10 marks)

(a) A telecommunications company is deploying a new fibre-to-the-node broadband network. Optical fibre carries the signal as far as a 'street cabinet' from which point the signal is transmitted to the customer over ordinary 'telephone wires'. The company claims that the average distance of a customer from a cabinet will be 300 m. What is the maximum proportion of customers that are located more than 1.5 km from a cabinet? (2 marks)

(b) Consider independent normal random variables X and Y, each with mean 0 and variance σ^2. Show that, if $R = \sqrt{X^2 + Y^2}$, then R has a Rayleigh distribution, *i.e.*, R has the p.d.f.

$$f_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right).$$

(2 marks)

(c) Recall that a stochastic process is a function $X(t, \omega)$ where t represents time and ω represents an outcome. In this context, define the relationship with:

i) a realisation,

ii) the ensemble and

iii) the state.

(3 marks)

(d) Consider a simple communication system that, at the n^{th} bit interval, transmits either a 1 or a 0 with equal probability, from the receiver's point of view. Show that this stochastic process is discrete-time wide-sense-stationary white noise. (3 marks)

Question 4. (10 marks)

(a) A homogeneous discrete Markov chain is specified by the following initial probability vector and transition probability matrix

$$P(0) = [0.2, 0.1, 0.3, 0.25, 0.15]$$

$$P = \begin{bmatrix} 0.3 & 0 & 0.3 & 0.1 & 0.3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

i) Draw a graph to represent this Markov chain, labelling all edges and denoting states as $s_1, \ldots, s_5$. (3 marks)

ii) Is the Markov chain ergodic? Explain briefly why/why not. (2 marks)

(b) The Metropolis-Hastings (MCMC) algorithm can be written as follows:

1. Initialize $x(0)$
2. For $i = 0$ to $N - 1$

   A. Generate a random number with uniform distribution over $(0, 1)$:

   $$u \sim \mathcal{U}_{[0,1]}$$

   B. Sample $x^* \sim q(x^*|x(i))$

   C. If $u < A(x(i), x^*) = \min\left(1, \dfrac{\pi_x q(x^*|x))}{\pi_{x^*} q(x|x^*))}\right)$

   $$x(i+1) = x^*$$

   Else

   $$x(i+1) = x(i)$$

Explain each of the different possible outcomes of Step 2C of the algorithm and hence how this step follows the basic principle of Markov Chain Monte Carlo sampling (*i.e.*, to draw samples for a target distribution of interest). (5 marks)

Question 5. (10 marks)

(a) Consider the function $f(\mathbf{x}) = x_2^2 + x_1x_2 + 3x_2 + 2x_1 + 3$.

i) Calculate the gradient of this function, $\nabla f(\mathbf{x})$. (2 marks)

ii) Find the critical point $\mathbf{x}^1$ of $f(\mathbf{x})$. Calculate the Hessian matrix, $\mathbf{H}_f$, and evaluate it at this critical point. (3 marks)

iii) It can be shown that $\mathbf{H}_f(\mathbf{x}^1)$ is indefinite. What type of critical point is $\mathbf{x}^1$? (1 mark)

(b) Consider the constrained optimization problem

Minimize

$$f(\mathbf{x}) = 5x_1^2 + 2x_2^2$$

subject to

$$h_1(\mathbf{x}) = x_2 - x_1 + 2 \leq 0,$$

$$h_2(\mathbf{x}) = x_1 + x_2 + 2 \leq 0.$$

i) Write down the Lagrangian function for this problem. (1 mark)

ii) Draw a diagram showing the constraints and contours of the objective function in the (x_1, x_2) solution space. Indicate by shading the feasible region of the space. (3 marks)

Question 6. (10 marks)

(a) i) Golden Section Search is an algorithm that is often used for optimizing a one-dimensional unimodal function. What is the convergence rate of Golden Section Search? (1 mark)

ii) The Golden Section Search algorithm is shown in Table 1. Given the starting values $a = -5, b = 1$ and assuming that the function to be optimized $f(x) = 2x^2 - 2$ is unimodal on the interval $[a, b]$, calculate the values of a, b, x_1, f_1, x_2 and f_2 after the first iteration of the algorithm (with the condition on the while loop assumed to be true). (4 marks)

(b) Write down pseudocode for the steepest descent algorithm for unconstrained minimization, explaining any notation you use. (3 marks)

(c) In two or three sentences, explain why a line search is typically used to determine the value of the step size parameter in the steepest descent algorithm. (2 marks)

Table 1: Golden Section Search algorithm for Question 6(a).

```
begin
  τ = (√(5) − 1)/2
  x1 = a + (1 − τ)(b − a)
  f1 = f(x1)
  x2 = a + τ(b − a)
  f2 = f(x2)
  while ((b − a) > tol) do
        if (f1 > f2) then
          a = x1
          x1 = x2
          f1 = f2
          x2 = a + τ(b − a)
          f2 = f(x2)
        else
          b = x2
          x2 = x1
          f2 = f1
          x1 = a + (1 − τ)(b − a)
          f1 = f(x1)
        end
  end
end
```

Question 7. (10 marks)

(a) In general, global optimization is considered to be a very difficult problem. Briefly discuss one reason for this. You may use an example or draw a diagram to assist in your explanation. (5 marks)

(b) In the Simulated Annealing, a move from the current point in the search space, $\mathbf{x}_c$ to a proposed new point $\mathbf{x}_n$ is determined according to the following rule (for minimization):

if $f(\mathbf{x}_n) < f(\mathbf{x}_c)$ **then**
  $\mathbf{x}_c = \mathbf{x}_n$
else
  if $random[0,1) < e^{\frac{f(\mathbf{x}_c)-f(\mathbf{x}_n)}{T}}$ **then**
    $\mathbf{x}_c = \mathbf{x}_n$

Explain the effect of varying the value of the temperature parameter T on the behaviour of the algorithm search trajectory. (5 marks)

Question 8. (10 marks)

(a) In the area of Metaheuristic Optimization, "intensification" and "diversification" have been suggested as key concepts in the analysis and development of more powerful algorithms for hard optimization problems. Give a brief explanation of what these two terms mean. For any one of the metaheuristic algorithms discussed in lectures, explain how the operators in the algorithm relate to intensification and diversification. (6 marks)

(b) In a Travelling Salesman Problem (TSP), a straightforward way of representing a candidate solution (tour) is as a vector of integers, representing the order of cities visited (where cities are numbered in some way). For example,

$$S_1 = [1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 1]$$

represents a valid tour for an eight-city TSP. However, when applying an Evolutionary algorithm, producing a crossover operator with this representation is not straightforward. To demonstrate this, consider the 2-point crossover operator (discussed in lectures for binary solution vectors). Give an example (using S_1 above and another valid solution) and explain one difficulty in applying 2-point crossover to this problem representation. (4 marks)