THE UNIVERSITY OF QUEENSLAND

VENUE:
SEAT NUMBER:
STUDENT NUMBER:

FINAL EXAMINATION

St Lucia Campus

Semester One 2009

ENGG7302 ADVANCED COMPUTATIONAL TECHNIQUES IN ENGINEERING

PERUSAL TIME	**10 mins. During perusal, write on the blank paper provided**
WRITING TIME	**3:00 Hours**
EXAMINER	**A/Prof. Vaughan Clarkson & Dr. Marcus Gallagher**

This examination paper has 7 pages (*include title page and attachments*) and is printed on Double-Sided

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Exam Type:	**Closed Book - Specified materials permitted**
Permitted Materials:	**Calculator - Yes - EPSA approved (must have label)** **Dictionary - Yes - Unmarked paper Bilingual Dictionary only** **Other -** **– No electronic aids are permitted (e.g. laptops, phone)**
Answer: (*Where students should write answers)*	**All Questions on Writing booklet** ______ ______________ ______ ______________ ______ ______________
Other Instructions:	
Total Number of Questions: ***(for the whole examination)***	**8**
Total Number of Marks and Overall % Weighting	**80 total number of marks** **50 overall % weighting**

Students must comply with the General Award Rules 1A.5 and 1A.7 which outline the responsibilities of students during an examination.

Question 1. (10 marks)

(a) Define what is meant by:

i) a Vandermonde matrix, (1 mark)

ii) a Toeplitz matrix. (1 mark)

(b) Show that the definition of the Frobenius norm so that

$$\|\mathbf{A}\|_{\mathrm{F}} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2}$$

is equivalent to the definition

$$\|\mathbf{A}\|_{\mathrm{F}} = \sqrt{\mathrm{tr}(\mathbf{A}^H\mathbf{A})}.$$ (2 marks)

(c) Calculate the reduced SVD of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ j & j \\ 0 & 1 \end{pmatrix}.$$ (6 marks)

Question 2. (10 marks)

(a) Suppose the matrix $\mathbf{Q}$ is Hermitian and unitary. Show that $\mathbf{P} = \frac{1}{2}(\mathbf{I} - \mathbf{Q})$ is a projection matrix. (2 marks)

(b) A matrix $\mathbf{A}$ has the reduced QR decomposition $\mathbf{A} = \hat{\mathbf{Q}}\hat{\mathbf{R}}$ where

$$\hat{\mathbf{Q}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{R}} = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix}.$$

Compute a full QR decomposition. (2 marks)

(c) The *LQ decomposition* of a matrix $\mathbf{A}$ is the decomposition of $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{L}\mathbf{Q}$$

where $\mathbf{Q}$ is a unitary matrix and $\mathbf{L}$ is a *lower triangular* matrix, *i.e.*, $\mathbf{L}^H$ is upper triangular. Write a MATLAB function that calculates the LQ decomposition of a matrix. Your code may make use of MATLAB's `qr` function. (3 marks)

(d) Consider an experiment in which the object is to measure the acceleration of a rocket at take-off. At $t = 0$, the rocket's engines ignite and the rocket is stationary at position $y = 0$. Measurements are made at times $t_1, \ldots, t_n$ of the position $y_1, \ldots, y_n$, during which time it is assumed that the acceleration is constant. Derive a formula for the least-squares estimate of acceleration. (3 marks)

Question 3. (10 marks)

(a) Consider a pair of discrete r.v.s X and Y for which the joint p.m.f. is

$$f_{X,Y}[x,y] = \begin{cases} \frac{1}{3} & \text{if } (x,y) = (0,0) \text{ or } (x,y) = (0,1) \\ \frac{1}{6} & \text{if } (x,y) = (1,0) \text{ or } (x,y) = (1,1) \\ 0 & \text{otherwise.} \end{cases}$$

Determine:

i) the marginal p.m.f. $f_X[x]$, (1 mark)

ii) whether X and Y are independent and (1 mark)

iii) the conditional p.m.f. $f_{Y|X=1}[y]$. (1 mark)

(b) For a sequence of i.i.d. r.v.s $X_1, X_2, \ldots$ with

$$S_n = \sum_{i=1}^{n} X_i \qquad \text{and} \qquad \bar{X}_n = \frac{S_n}{n}$$

state:

i) the weak law of large numbers and (1 mark)

ii) the central limit theorem. (1 mark)

(c) Let $X(t) = ae^{j(\Omega t+\Phi)}$ where a is a constant and Ω and Φ are independent r.v.s. Suppose Φ has a uniform distribution over 2π radians and Ω has an arbitrary p.d.f. $f_\Omega(\omega)$.

i) Show that $X(t)$ is WSS. (3 marks)

ii) Hence show that the PSD of $X(t)$ is

$$S_X(\omega) = 2\pi a^2 f_\Omega(\omega).$$ (2 marks)

Hint: Recall that, when using angular frequency ω, a function $y(t)$ and its Fourier transform $Y(\omega)$ are related by the inverse Fourier transform

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}\,d\omega.$$

Question 4. (10 marks)

(a) Company A and Company B both manufacture and sell very good beer, and they are the only companies which sell beer in a given small town. Each year the market in this town will absorb exactly 4 000 barrels of beer. Company A has an advertising gimmick which it thinks in any given year will increase sales by 1 000 barrels of beer with probability $\frac{3}{5}$ and decrease sales by 1 000 barrels with probability $\frac{2}{3}$, except that once a company's sales are down to 0, it gives up its business in this town.

i) Define the states for a Markov chain which will describe this situation. (1 mark)

ii) Write down the transition probability matrix for this chain. (2 marks)

iii) Does the state space of this Markov chain form a communicating class? Explain briefly why/why not. (2 marks)

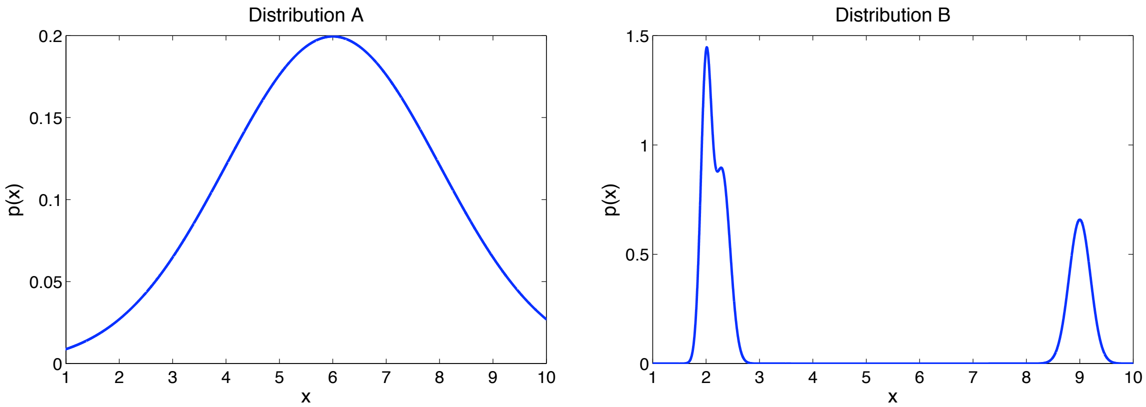

Figure 1: Probability density functions for Question 4(b).

(b) Consider the two probability density functions shown in Figure 1.

An engineer is considering applying the Metropolis-Hastings (MCMC) algorithm to draw samples from each of these distributions. She intends to use the Gaussian proposal distribution: $\mathcal{N}(0, 0.1)$.

Describe how you expect the algorithm will perform on these distributions. Your answer should mention the acceptance probability and the mixing of the chains and should compare the behaviour of the algorithm on the two problems. (5 marks)

Question 5. (10 marks)

(a) Consider the function $f(\mathbf{x}) = 2x_1^3 + 3x_1^2 + 7x_1x_2x_3 + x_2^2 + x_3^3 + 4$.

i) Calculate the gradient, $\nabla f(\mathbf{x})$, and Hessian, $\mathbf{H}_f$, of this function. (3 marks)

ii) The steepest descent algorithm is to be applied to minimize $f(\mathbf{x})$. Assuming a starting point $\mathbf{x}_0 = [1\ 1\ 1]^T$ and fixed step size parameter $\alpha = 0.1$, carry out one iteration of the algorithm and find the value of the next search point $\mathbf{x}_1$ and its objective function value. (3 marks)

iii) Newton's method is to be applied to minimize $f(\mathbf{x})$. Assuming a starting point $\mathbf{x}_0 = [1\ 1\ 1]^T$, calculate the linear system to be solved to find the Newton step (you do not need to solve this system). (2 marks)

iv) The Newton step from iii) above is approximately $\mathbf{s}_0 = [-0.74\ \ 0.05\ \ -0.86]^T$. Carry out one iteration of the algorithm and find the value of the next search point $\mathbf{x}_1$ and its objective function value. (2 marks)

Table 1: Golden Section Search algorithm for Question 6(a).

begin
  $\tau = (\sqrt(5) - 1)/2$
  $x_1 = a + (1 - \tau)(b - a)$
  $f_1 = f(x_1)$
  $x_2 = a + \tau(b - a)$
  $f_2 = f(x_2)$
  while $((b - a) > tol)$ **do**
    if $(f_1 > f_2)$ **then**
      $a = x_1$
      $x_1 = x_2$
      $f_1 = f_2$
      $x_2 = a + \tau(b - a)$
      $f_2 = f(x_2)$
    else
      $b = x_2$
      $x_2 = x_1$
      $f_2 = f_1$
      $x_1 = a + (1 - \tau)(b - a)$
      $f_1 = f(x_1)$
    end
  end
end

Question 6. (10 marks)

(a) i) Successive parabolic interpolation is an algorithm that is often used for optimising a one-dimensional unimodal function. Is the convergence rate of this algorithm linear, superlinear or quadratic? (1 mark)

ii) The Golden Section Search algorithm is shown in Table 1. Given the starting values $a = -1, b = 3$ and assuming that the function to be optimized $f(x) = 2x^2 - 2$ is unimodal on the interval $[a, b]$, calculate the values of a, b, x_1, f_1, x_2 and f_2 after the first iteration of the algorithm (with the condition on the while loop assumed to be true). (4 marks)

(b) Show that the following optimisation problem has no feasible solution:

$$\min_{\mathbf{x}} f(\mathbf{x}) = 0.5x_1^2 + 2.5x_2^2$$

subject to

$$h_1(\mathbf{x}) = x_2 - x_1 + 1 \leq 0,$$

$$h_2(\mathbf{x}) = x_2 - x_1 - 1 \leq 0.$$

(3 marks)

(c) In two or three sentences, explain why a line search is typically used to determine the value of the step size parameter in the steepest descent algorithm. (2 marks)

Question 7. (10 marks)

(a) Metaheuristic optimisers are *approximate* algorithms in the sense that they "…sacrifice the guarantee of finding optimal solutions for the sake of getting good solutions in a significantly reduced amount of time." In a few sentences, discuss why this approach makes sense from the point of view of global optimisation. (5 marks)

(b) Explain the relationship between the Simulated Annealing (optimisation) algorithm and the Metropolis-Hastings (MCMC) algorithm. Your answer should highlight the similarities and differences between the steps in each algorithm and how the algorithms are used. You may draw a diagram or use pseudocode to assist in your explanation. (5 marks)

Question 8. **(10 marks)**

(a) In the area of Metaheuristic Optimisation, “intensification” and “diversification” have been suggested as key concepts in the analysis and development of more powerful algorithms for hard optimisation problems. Give a brief explanation of what these two terms mean. For any one of the metaheuristic algorithms discussed in lectures, explain how the operators in the algorithm relate to intensification and diversification. (4 marks)

(b) In Genetic Algorithms, many different kinds of operators have been proposed. For discrete solutions, the so-called *insert mutation* operator works by selecting at random two of the genes (positions) in a solution vector, and moving one so that it is next to the other, shuffling any genes in between along to make room.

i) Given the solution vector [1 2 3 4 5 6 7 8], apply an insert mutation, assuming that genes 2 and 5 are the two genes chosen at random. Also assume that the shuffling of the “in between” genes moves them from left to right. (2 marks)

ii) Given the previous assumptions, calculate the probability that an insert mutation will result in a reordering of 7 genes in a solution vector, assuming a solution vector of length 8. (2 marks)

(c) Calculate the probability that, in a Genetic Algorithm, a binary solution vector with length L will not be changed by applying the usual bit-flip mutation with probability $p_{\mathrm{m}} = 1/L$. (2 marks)

