THIS PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

STUDENT NAME:
STUDENT NUMBER:

Internal Students Only

THE UNIVERSITY OF QUEENSLAND

School of Information Technology & Electrical Engineering

Second Semester Examination, November 2007

ENGG7302 / ELEC4002

ADVANCED COMPUTATIONAL TECHNIQUES IN ENGINEERING

(M.E. / B.E. IV)

CLOSED BOOK

TIME: **THREE** hours for working

TEN minutes for perusal before examination begins

ANSWER ALL QUESTIONS IN BOOKLET PROVIDED

QUESTIONS CARRY THE NUMBER OF MARKS INDICATED

Drawing instruments and one battery-operated or solar-powered electronic calculator may be used but NO pre-programmed material or calculator instruction booklets are allowed in the examination room.

Question 1. (10 marks)

(a) Show that $\|\mathbf{Qx}\| = \|\mathbf{x}\|$ when $\mathbf{Q}$ is a unitary matrix. (2 marks)

(b) Show that, for a rank-1 matrix, the 2-norm and Frobenius norm are equal. (2 marks)

(c) Calculate the full SVD of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(6 marks)

Question 2. (10 marks)

(a) Suppose the matrix $\mathbf{Q}$ is Hermitian and unitary. Show that $\mathbf{P} = \frac{1}{2}(\mathbf{I} - \mathbf{Q})$ is a projection matrix. (2 marks)

(b) Compute the full QR decomposition of the matrix of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \end{pmatrix}.$$

(3 marks)

(c) Consider the algorithm shown in Table 1 for computing the product $\mathbf{Qb}$ for an input vector $\mathbf{b}$ where $\mathbf{Q}$ is represented by the vectors $\boldsymbol{\phi}_k$ computed by Householder triangularisation.

Table 1: Algorithm for Question 2(c).

```
begin
  for k = n to 1 step -1 do
      β = b_{k:m}
      β = β − 2φ_k(φ_k^H β)
      b_{k:m} = β
  od
end
```

Determine the floating-point operation count for this algorithm.

Hint: Recall that $\displaystyle\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$. (3 marks)

(d) Briefly explain how the pseudo-inverse can be used to obtain the least-squares solution to a set of overdetermined linear equations. (2 marks)

Question 3. (10 marks)

(a) Consider a r.v. X with $\mu = 0$ and $\sigma^2 = 1$. Show that Chebychev's inequality implies that $F(x) \leq x^{-2}$ when $x < 0$. (2 marks)

(b) In a communications system, the transmitter attempts to communicate information with the receiver in the form of a series of *bits*. However, the receiver makes *bit errors* with a probability of 10^{-5}, *i.e.*, it has a *bit error rate (BER)* of 10^{-5}. Derive an expression for the probability that no bit errors occur at the receiver in a block of 10^5 bits. (3 marks)

(c) State the definition of *wide-sense stationarity*. (2 marks)

(d) The randomly phased sinusoid $X(t) = \cos(2\pi f_0 t + \Phi)$ is WSS when $f(1) = f(2) = 0$ and $f(\lambda) = E[e^{j\lambda\Phi}]$. Derive an expression for its PSD.

Hint: Recall that $e^{j2\pi f_0 \tau} \tilde{f} \delta(f - f_0)$. (3 marks)

Question 4. (10 marks)

(a) A homogeneous discrete Markov chain is specified by the following initial probability vector and transition probability matrix

$$P(0) = [0.2, 0.1, 0.3, 0.25, 0.15]$$

$$P = \begin{bmatrix} 0.3 & 0.4 & 0 & 0 & 0.3 \\ 0.3 & 0.4 & 0 & 0 & 0.3 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0.3 & 0.4 & 0 & 0 & 0.3 \end{bmatrix}$$

Draw a graph to represent this Markov chain, labelling all edges and denoting states as $s_1, \ldots, s_5$. (3 marks)

(b) Is the Markov chain from part **(a)** ergodic? Explain briefly why/why not. (2 marks)

(c) Markov chain Monte Carlo (MCMC) algorithms can be used to draw samples from a large class of distributions. Explain the basic working principal of MCMC algorithms (i.e the function of the equilibrium distribution of a Markov chain in a MCMC algorithm). (5 marks)

Question 5. (10 marks)

Consider the function $f(\mathbf{x}) = 5x_1^2 + x_2^3 + 3x_1x_2 + 5$.

(a) Calculate the gradient of this function, $\nabla f(\mathbf{x})$. (2 marks)

(b) $f(\mathbf{x})$ has two critical points at $\mathbf{x}^1 = [0\ 0]$ and $\mathbf{x}^2 = [-\frac{9}{50}\ \frac{3}{5}]$. Calculate the Hessian matrix, $\mathbf{H}_f$ and evaluate it at each of the critical points. (2 marks)

(c) It can be shown that $\mathbf{H}_f(\mathbf{x}^2)$ is positive definite, while $\mathbf{H}_f(\mathbf{x}^1)$ is not positive definite. What type of critical points are $\mathbf{x}^1$ and $\mathbf{x}^2$? (2 marks)

Consider the constrained optimization problem

Minimize

$$f(\mathbf{x}) = 3x_1^2 + x_2^2$$

Subject to

$$h_1(\mathbf{x}) = x_2 - x_1 + 2 \leq 0$$

$$h_2(\mathbf{x}) = x_1 + x_2 + 2 \leq 0$$

(d) Write down the Lagrangian function for this problem. (1 mark)

(e) Draw a diagram showing the constraints and contours of objective function in the (x_1, x_2) solution space. Indicate by shading the feasible region of the space. (3 marks)

Question 6. (10 marks)

(a) Golden Section Search is an algorithm that is often used for optimizing a one-dimensional unimodal function. What is the convergence rate of Golden Section Search? (1 mark)

(b) The Golden Section Search algorithm is shown in Table 2. Given the starting values $a = -2, b = 4$ and assuming that the function to be optimized $f(x) = 2x^2 - 2$ is unimodal on the interval $[a, b]$, calculate the values of a, b, x_1, f_1, x_2 and f_2 after the first iteration of the algorithm. (4 marks)

(c) Write down pseudocode for the steepest descent algorithm for unconstrained minimization, explaining any notation you use. (3 marks)

(d) In two or three sentences, explain why a line search is typically used to determine the value of the step size parameter in the steepest descent algorithm. (2 marks)

Table 2: Golden Section Search algorithm for Question 6(b).

```
begin
  τ = (√(5) − 1)/2
  x1 = a + (1 − τ)(b − a)
  f1 = f(x1)
  x2 = a + τ(b − a)
  f2 = f(x2)
  while ((b − a) > tol) do
        if (f1 > f2) then
          a = x1
          x1 = x2
          f1 = f2
          x2 = a + τ(b − a)
          f2 = f(x2)
        else
          b = x2
          x2 = x1
          f2 = f1
          x1 = a + (1 − τ)(b − a)
          f1 = f(x1)
        end
  end
end
```

Question 7. (10 marks)

(a) In general, global optimization is considered to be a very difficult problem. Briefly discuss one reason for this. You may use an example or draw a diagram to assist in your explanation. (5 marks)

(b) In the Simulated Annealing, a move from the current point in the search space, $\mathbf{x}_c$ to a proposed new point $\mathbf{x}_n$ is determined according to the following rule (for minimization):

if $f(\mathbf{x}_n) < f(\mathbf{x}_c)$ **then**
  $\mathbf{x}_c = \mathbf{x}_n$
else
  if $random[0,1) < e^{\frac{f(\mathbf{x}_c)-f(\mathbf{x}_n)}{T}}$ **then**
    $\mathbf{x}_c = \mathbf{x}_n$

Explain the effect of varying the value of the temperature parameter T on the behaviour of the algorithm search trajectory. (5 marks)

Question 8. (10 marks)

(a) In the area of Metaheuristic optimization, "intensification" and "diversification" have been suggested as key concepts in the analysis and development of more powerful algorithms for hard optimization problems. Give a brief explanation of what these two terms mean. For any one of the metaheuristic algorithms discussed in lectures, explain how the operators in the algorithm relate to intensification and diversification. (6 marks)

(b) In a Travelling Salesman Problem (TSP), a straightforward way of representing a candidate solution (tour) is as a vector of integers, representing the order of cities visited (where cities are numbered in some way). For example,

$$S_1 = [1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 1]$$

represents a valid tour for an eight-city TSP. However, when applying an Evolutionary algorithm, producing a crossover operator with this representation is not straightforward. To demonstrate this, consider the 1-point crossover operator (discussed in lectures for binary solution vectors). Give an example (using S_1 above and another valid solution) and explain one difficulty in applying 1-point crossover to this problem representation. (4 marks)

