THIS PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

STUDENT NAME:
STUDENT NUMBER:

Internal Students Only

THE UNIVERSITY OF QUEENSLAND

School of Information Technology & Electrical Engineering

Final Semester Examination, Semester Two 2008

ENGG7302

ADVANCED COMPUTATIONAL TECHNIQUES IN ENGINEERING

(M.E.)

CLOSED BOOK

TIME: **THREE** hours for working

TEN minutes for perusal before examination begins

ANSWER ALL QUESTIONS IN BOOKLET PROVIDED

QUESTIONS CARRY THE NUMBER OF MARKS INDICATED

Drawing instruments and one battery-operated or solar-powered electronic calculator may be used but NO pre-programmed material or calculator instruction booklets are allowed in the examination room.

Question 1. (10 marks)

(a) A permutation matrix is a matrix such that every row and every column contains precisely a single 1 with 0s everywhere else. Show that permutation matrices are unitary. (2 marks)

(b) Show that

$$\|\mathbf{A}\|_1 = \max_{i=1,\ldots,n} \|\mathbf{a}_i\|_1$$

(2 marks)

(c) Calculate the full SVD of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(6 marks)

Question 2. (10 marks)

(a) Consider the subspace in $\mathbb{C}^n$ that consists of vectors whose coordinates sum to zero. Calculate an orthogonal projection matrix that projects onto this subspace. (3 marks)

(b) Compute the full QR decomposition of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 1 & -1 \end{pmatrix}.$$

(3 marks)

(c) Consider a set of points on the plane $\{(x_i, y_i)\}$, $i = 1, \ldots, n$. The objective is to fit a straight line $y = mx + b$ to the points.

i) Show that this can be formulated as a least squares problem.

ii) Write an expression involving a pseudo-inverse that solves for the parameters m and b.

(4 marks)

Question 3. **(10 marks)**

(a) Let T be the time that elapses between successive packets arriving at an ethernet router. It is found that $E[T] = 0.2\,\mu$s.

i) Derive an upper bound on the probability that $T \geq 1\,\mu$s.

ii) Show that there is a distribution on T that achieves the upper bound.

(5 marks)

(b) Let $X(t) = ae^{j(\Omega t+\Phi)}$ where a is a constant and Ω and Φ are independent r.v.s. Suppose Φ has a uniform distribution over 2π radians and Ω has an arbitrary p.d.f. $f_\Omega(\omega)$.

i) Show that $X(t)$ is WSS.

ii) Hence show that the PSD of $X(t)$ is

$$S_X(\omega) = 2\pi a^2 f_\Omega(\omega).$$

Hint: Recall that, when using angular frequency ω, a function $y(t)$ and its Fourier transform $Y(\omega)$ are related by the inverse Fourier transform

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}\,d\omega.$$

(5 marks)

Question 4. (10 marks)

(a) Six boys (Dick, Harry, Joe, Mark, Sam and Tom) play catch. If Dick has the ball, he is equally likely to throw it to Harry, Mark, Sam or Tom. If Harry gets the ball, he is equally likely to throw it to Dick, Joe, Sam or Tom. If Sam has the ball, he is equally likely to throw it to Dick, Harry, Mark or Tom. If any one of Joe or Tom gets the ball, they keep throwing it to each other. If Mark gets the ball, he runs away with it.

i) Write down the transition probability matrix for an appropriate Markov chain to model this game. If you need to make any assumptions about the problem, state them clearly. (2 marks)

ii) Classify each of the states in the chain as either recurrent or transient. (2 marks)

(b) The Metropolis-Hastings (MCMC) algorithm can be written as follows:

1. Initialize $x(0)$
2. For $i = 0$ to $N - 1$

   A. Generate a random number with uniform distribution over $(0, 1)$:

   $$u \sim \mathcal{U}[0, 1]$$

   B. Sample $x^* \sim q(x^*|x(i))$

   C. If $u < A(x(i), x^*) = \min\left(1, \frac{\pi_x q(x^*|x))}{\pi_{x^*} q(x|x^*))}\right)$

   $$x(i + 1) = x^*$$

   Else

   $$x(i + 1) = x(i)$$

Assume that the proposal distribution $q(x^*|x(i))$ is Gaussian. In terms of the acceptance probability and mixing of the chain:

i) Explain briefly the expected behavior of the dynamics of the chain if the proposal distribution is well-matched to the target distribution. (2 marks)

ii) Explain what might be observed if the variance of q is too large. (2 marks)

iii) Explain what might be observed if the variance of q is too small. (2 marks)

Question 5. (10 marks)

(a) Consider the function $f(\mathbf{x}) = 100(x_2 - x_1^2)^2 + (1 - x_2)^2$.

i) Calculate the gradient, $\nabla f(\mathbf{x})$ and Hessian, $\mathbf{H}_f$ of this function. (3 marks)

ii) Verify that the point $\mathbf{x}^* = (1, 1)^T$ satisfies $\nabla f(\mathbf{x}) = \mathbf{0}$ and evaluate $\mathbf{H}_f(\mathbf{x}^*)$. (2 marks)

iii) The eigenvalues of $\mathbf{H}_f(\mathbf{x}^*)$ are (approximately) $\lambda_1 = 0.0762$ and $\lambda_2 = 209.9238$. What does this prove about the point $\mathbf{x}^*$? (1 mark)

(b) Given the objective function

$$f(x) = 5x^6 - 3x^4 + \frac{1}{2}x^2$$

Carry out one iteration of Newton's method, starting from the point $x_0 = 0.5$. Show all of your working. (4 marks)

Question 6. (10 marks)

(a) i) Successive parabolic interpolation is an algorithm that is often used for optimizing a one-dimensional unimodal function. Is the convergence rate of this algorithm linear, superlinear or quadratic? (1 mark)

ii) The Golden Section Search algorithm is shown in Table 1. Given the starting values $a = -1, b = 4$ and assuming that the function to be optimized $f(x) = 2x^2 - 2$ is unimodal on the interval $[a, b]$, calculate the values of a, b, x_1, f_1, x_2 and f_2 after the first iteration of the algorithm (with the condition on the while loop assumed to be true). (4 marks)

(b) Write down pseudocode for the steepest descent algorithm for unconstrained minimization, explaining any notation you use. (3 marks)

(c) In two or three sentences, explain why a line search is typically used to determine the value of the step size parameter in the steepest descent algorithm. (2 marks)

Table 1: Golden Section Search algorithm for Question 6(a).

begin
  $\tau = (\sqrt(5) - 1)/2$
  $x_1 = a + (1 - \tau)(b - a)$
  $f_1 = f(x_1)$
  $x_2 = a + \tau(b - a)$
  $f_2 = f(x_2)$
  while $((b - a) > tol)$ **do**
      if $(f_1 > f_2)$ **then**
        $a = x_1$
        $x_1 = x_2$
        $f_1 = f_2$
        $x_2 = a + \tau(b - a)$
        $f_2 = f(x_2)$
      else
        $b = x_2$
        $x_2 = x_1$
        $f_2 = f_1$
        $x_1 = a + (1 - \tau)(b - a)$
        $f_1 = f(x_1)$
      end
  end
end

Question 7. (10 marks)

(a) Metaheuristic optimizers are *approximate* algorithms in the sense that they "...sacrifice the guarantee of finding optimal solutions for the sake of getting good solutions in a significantly reduced amount of time." In a few sentences, discuss why this approach makes sense from the point of view of global optimization. (5 marks)

(b) In Simulated Annealing, a move from the current point in the search space, $\mathbf{x}_c$, to a proposed new point, $\mathbf{x}_n$, is determined according to the following rule (for minimization):

if $f(\mathbf{x}_n) < f(\mathbf{x}_c)$ **then**
  $\mathbf{x}_c = \mathbf{x}_n$
else
  if $random[0,1) < e^{\frac{f(\mathbf{x}_c) - f(\mathbf{x}_n)}{T}}$ **then**
    $\mathbf{x}_c = \mathbf{x}_n$

Explain the effect of varying the value of the temperature parameter, T, on the behaviour of the algorithm's search trajectory. (5 marks)

Question 8. (10 marks)

(a) In the area of Metaheuristic Optimization, "intensification" and "diversification" have been suggested as key concepts in the analysis and development of more powerful algorithms for hard optimization problems. Give a brief explanation of what these two terms mean. For any one of the metaheuristic algorithms discussed in lectures, explain how the operators in the algorithm relate to intensification and diversification. (4 marks)

(b) In Genetic Algorithms, many different kinds of operators have been proposed. For discrete solutions, the so-called *insert mutation* operator works by selecting at random two of the genes (positions) in a solution vector, and moving one so that it is next to the other, shuffling any genes in between along to make room.

i) Given the solution vector [1 2 3 4 5 6 7 8], apply an insert mutation, assuming that genes 3 and 6 are the two genes chosen at random. Also assume that the shuffling of genes in between moves genes from left to right. (2 marks)

ii) Given the previous assumptions, calculate the probability that an insert mutation will result in a rearrangement of every single gene in a solution vector, assuming a solution vector of length 8. (2 marks)

(c) Calculate the probability that in a Genetic Algorithm, a binary solution vector with length L will not be changed by applying the usual bit-flip mutation with probability $p_m = 1/L$. (2 marks)

 END OF EXAM