THIS PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

STUDENT NAME:
STUDENT NUMBER:

Internal Students Only

THE UNIVERSITY OF QUEENSLAND

School of Information Technology & Electrical Engineering

Final Semester Examination, Semester Two 2010

ENGG7302

ADVANCED COMPUTATIONAL TECHNIQUES IN ENGINEERING

(M.E.)

CLOSED BOOK

TIME: **THREE** hours for working

TEN minutes for perusal before examination begins

ANSWER ALL QUESTIONS IN BOOKLET PROVIDED

ALL QUESTIONS HAVE EQUAL VALUE

EAIT approved and labelled calculators only.

Question 1. (10 marks)

(a) The *Schlatten p-norm* of a matrix $\mathbf{A}$ of rank r is defined so that

$$\|\mathbf{A}\|_p = \left(\sum_{i=1}^{r} \sigma_i^p \right)^{1/p}$$

where $\sigma_1, \ldots, \sigma_r$ are the singular values of $\mathbf{A}$.

i) As $p \to \infty$, to which other well-known matrix norm does the Schlatten p-norm converge? (1 mark)

ii) To which other well-known matrix norm is the Schlatten 2-norm equivalent? (1 mark)

iii) The Schlatten 1-norm is also known as the *trace norm*. Why? (1 mark)

iv) The Schlatten p-norm is not truly a matrix norm for $p < 1$. Nevertheless, as $p \to 0$, to which other well-known matrix operation does $\|\mathbf{A}\|_p^p$ converge? (1 mark)

(b) Calculate the reduced SVD of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

(6 marks)

Question 2. (10 marks)

(a) Show that, when $\mathbf{P}$ is a projection matrix, $e^{\mathbf{P}} = \mathbf{I} + (e-1)\mathbf{P}$. (3 marks)

(b) Given that the QR decomposition of

$$\mathbf{A} = \begin{pmatrix} 5 & 8 & 10 & -5 \\ 5 & 4 & 6 & -5 \\ 5 & 8 & 6 & -9 \\ 5 & 4 & 2 & 15 \end{pmatrix} \quad \text{has} \quad \mathbf{Q} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 10 & 12 & 12 & -2 \\ 0 & 4 & 4 & -12 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 0 & 12 \end{pmatrix},$$

compute the QR decomposition of

$$\mathbf{A}' = \begin{pmatrix} 5 & 8 & -10 & -5 \\ 5 & 4 & -6 & -5 \\ 5 & 8 & -6 & -9 \\ 5 & 4 & -2 & 15 \end{pmatrix}.$$

(3 marks)

(c) Consider an experiment in which the object is to measure the initial position and velocity of a vehicle along a straight track. Measurements are made at times $t_1, \ldots, t_n$ of the positions $x_1, \ldots, x_n$, during which time it is assumed that the velocity is constant.

i) Derive a formula, in matrix-vector notation, for the least-squares estimate of acceleration. (2 marks)

ii) Write a few lines of MATLAB to calculate the estimate given vectors `t` and `x`. (2 marks)

Question 3. (10 marks)

Let X and Y be two random variables. Their joint probability density function is given by

$$f_{xy}(x,y) = \begin{cases} 0.5 & \text{for} \quad y+x \geq 1 \text{ and } y-x \geq -1 \text{ and} \\ & \quad\quad\; y+x \leq 3 \text{ and } y-x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

as illustrated in Figure 1.

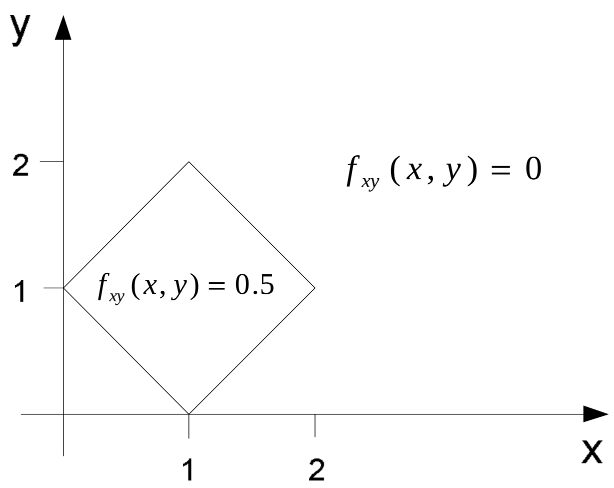

Figure 1: Joint probability density function $f_{xy}(x, y)$.

(a) Calculate the marginal probability density functions $f_x(x)$ and $f_y(y)$ of X and Y respectively. (4 marks)

(b) Using their marginal distributions, show whether the random variables X and Y are uncorrelated or not. (3 marks)

(c) Using their marginal distributions, show whether the random variables X and Y are independent or not. (1 mark)

(d) Let two random variables A and B be independent. Prove that A and B are also uncorrelated. (2 marks)

Question 4. **(10 marks)**

(a) Let X be a continuous random variable that is uniformly distributed in the interval $[-1;\ 1]$.
Let $Y = g(X) = X^2$.

Determine:

i) The cumulative distribution function $F_X(x)$ of X, (1 mark)

ii) The probability density function $f_Y(y)$ of Y, (3 marks)

iii) The cumulative distribution function $F_Y(y)$ of Y. (1 mark)

(b) A wide-sense stationary (WSS) process $X(t)$ is passed through the linear time-invariant (LTI) filter depicted in Figure 2. This LTI filter contains a time delay element in the lower path and transforms the input process X(t) into the output process Y(t).

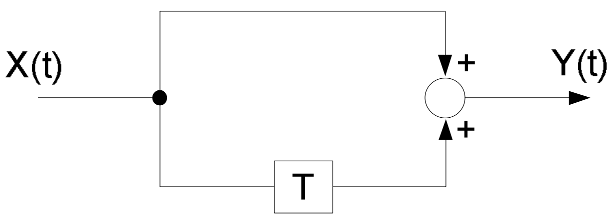

Figure 2: LTI filter.

i) State the definition of wide-sense stationarity. (2 marks)

ii) Derive the Power Spectral Density (PSD) $S_Y(f)$ of the output process $Y(t)$ as a function of the unknown PSD $S_X(f)$ of the input process $X(t)$. (3 marks)

Question 5. (10 marks)

(a) Consider a simple Markov chain weather model assuming that on any given day it is either fine (F, state 1), raining (R, state 2) or snowing (S, state 3). Assume that the model has the transition probability matrix

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.4 & 0.4 \\ 0.4 & 0.2 & 0.2 \end{pmatrix}$$

Calculate the probability that it will be fine in two days time given that it is snowing today. (3 marks)

(b) Consider the Markov chain with states $\{e_1, e_2, e_3, e_4\}$ and the transition probability matrix

$$P = \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.2 & 0.1 \end{pmatrix}$$

i) Is the chain aperiodic? (1 mark)

ii) Does the chain have any transient states (which one(s))? (1 mark)

iii) Does the chain have any absorbing states (which one(s))? (1 mark)

iv) Is the chain ergodic? (1 mark)

(c) Smith is in jail and has \$3; he can get out on bail if he has \$6. A guard agrees to make a series of bets with him based on the toss of a fair coin. If Smith loses all of his money, the game is over and he stays in jail. At a given time step, if Smith bets \$1 he wins back \$2 (i.e. his bet plus \$1) and if Smith bets \$2 he wins back \$4 (i.e. his bet plus \$2). At each time step, Smith chooses to bet \$1 or \$2 with equal probability, except for when his total money is either \$1 or \$5, when he only bets \$1.

Write down a suitable transition probability matrix and initial probability vector to model this scenario. (3 marks)

Question 6. (10 marks)

(a) Six boys (Dick, Harry, Joe, Mark, Sam and Tom) play catch. If Dick has the ball, he is equally likely to throw it to Harry, Mark, Sam or Tom. If Harry gets the ball, he is equally likely to throw it to Dick, Joe, Sam or Tom. If Sam has the ball, he is equally likely to throw it to Dick, Harry, Mark or Tom. If any one of Joe or Tom gets the ball, they keep throwing it to each other. If Mark gets the ball, he runs away with it.

i) Write down the transition probability matrix for an appropriate Markov chain to model this game. If you need to make any assumptions about the problem, state them clearly. (2 marks)

ii) Classify each of the states in the chain as either recurrent or transient. (2 marks)

(b) The Metropolis-Hastings (MCMC) algorithm can be written as follows:

1. Initialize $x(0)$
2. For $i = 0$ to $N - 1$

   A. Generate a random number with uniform distribution over $(0, 1)$:

   $$u \sim \mathcal{U}[0, 1]$$

   B. Sample $x^* \sim q(x^*|x(i))$

   C. If $u < A(x(i), x^*) = \min\left(1, \frac{\pi_x q(x^*|x))}{\pi_{x^*} q(x|x^*))}\right)$

   $$x(i + 1) = x^*$$

   Else

   $$x(i + 1) = x(i)$$

Assume that the proposal distribution $q(x^*|x(i))$ is Gaussian. In terms of the acceptance probability and mixing of the chain:

i) Explain briefly the expected behavior of the dynamics of the chain if the proposal distribution is well-matched to the target distribution. (2 marks)

ii) Explain what might be observed if the variance of q is too large. (2 marks)

iii) Explain what might be observed if the variance of q is too small. (2 marks)

Question 7. (10 marks)

(a) A wire carrying an electric current is surrounded by rubber insulation of outer radius r. The resistance of the wire generates heat, which is conducted through the insulation and convected into the surrounding air. The temperature of the wire can be shown to be

$$T = \frac{q}{2\pi}\left(\frac{\ln(r/a)}{k} + \frac{1}{hr}\right) + T_\infty$$

where

q = rate of heat generation in wire = 50 W/m

a = radius of wire = 5mm

k = thermal conductivity of rubber = 0.16 W/m.K

h = convective heat-transfer coefficient = 20 W/m^2.K

T_∞ = ambient temperature = 280K

Find r that minimizes T (Hint: recall that $\frac{d}{dx}\ln x = \frac{1}{x}, x > 0$). (3 marks)

(b) Consider the constrained optimization problem

Minimize

$$f(\mathbf{x}) = x_1^2 + x_2^2,\ \mathbf{x} \in \mathbb{R}^2$$

Subject to

$$h_1(\mathbf{x}) = x_2 - x_1^3 + 1 \geq 0$$

and the three points $\mathbf{x}^a = (0,0), \mathbf{x}^b = (1,0), \mathbf{x}^c = (-1,-4)$.

i) Which of the three points are infeasible? (2 marks)

ii) Show that one of the three points is a unique global minimum of $f(\mathbf{x})$. (2 marks)

(c) An example Matlab implementation of the successive parabolic interpolation algorithm is shown in Table 1. In this implementation, it is assumed that r and s are the lower and upper bounds respectively for the search, and that the objective function, f, is unimodal in this range. The code will also fail if $fr = fs = ft$ on any iteration of the algorithm. Explain how you would modify this code to check this assumption and failure condition and to "break" and print an error message if there is a problem. Write your modified Matlab function, but you do not need to rewrite any lines of code that are unchanged (just refer to them as, e.g. "original lines 2-4"). (3 marks)

Table 1: Matlab successive parabolic interpolation function for Question 7(c).

```
function x=spi(f,r,s,t,k)
x(1)=r;x(2)=s;x(3)=t;
fr=f(r);fs=f(s);ft=f(t);
for i=4:k+3
  x(i)=(r+s)/2-(fs-fr)*(t-r)*(t-s)/(2*((s-r)*(ft-fs)-(fs-fr)*(t-s)));
  t=s;s=r;r=x(i);
  ft=fs;fs=fr;fr=f(r);             % single function evaluation
end
```

Question 8. (10 marks)

(a) There are three ways that we can solve a multiple objective optimisation problem: *a priori*, *progressive* and *a posteriori*. Explain what each of these approaches mean with regards to the design of an optimisation algorithm and its relationship to the decision maker. (3 marks)

(b) Explain how a Genetic Algorithm Tournament Selection routine works (you can use a diagram in your explanation). (2 marks)

(c) Explain how the heuristic exponent parameter (β) can be used to control the intensification or diversification behaviour of the Ant Systems algorithm. (2 marks)

(d) The individual particles within a Particle Swarm Optimisation algorithm use a combination of influences (momentum, local best, global best) to determine their movement through the search space. Explain how you would expect the algorithm to behave if the particles used only momentum and local influence to guide their movement. (3 marks)