Advanced Computational Techniques in Engineering

Lecture SP02: *Random Variables*

This lecture:

1. Random Variables.
2. Functions of One Random Variable.

Ref: STAT2202 course notes, PP ch. 4–5.

Random Variables

A *random variable (r.v.)* X is actually a function from the probability space into $\mathbb{R}$, *i.e.*, $X : \Omega \mapsto \mathbb{R}$.

- Formally, we write $X(\omega)$, but usually abbreviate to just X.
- If the range $\Omega_X \subseteq \mathbb{R}$ of X is discrete, X is a *discrete r.v.*
- Otherwise, it is a *continuous r.v.*
- The range of X is also measurable.
  - That is, we can define events in terms of X, *e.g.*, the event $X \leqslant 4$ or $X \in \{1, 2, 5\}$, and calculate their probability.

Distribution & Mass Functions

We define the *(cumulative) distribution function (c.d.f.)* $F_X : \mathbb{R} \mapsto \mathbb{R}$ of a r.v. X as

$$F_X(x) = P(X \leqslant x).$$

- For discrete r.v.s, the c.d.f. is a 'staircase'.

- Properties of a c.d.f.:
  1. $\lim_{x \to -\infty} F_X(x) = 0$,
  2. $\lim_{x \to \infty} F_X(x) = 1$ and
  3. $F_X(x)$ is non-decreasing.
- For a discrete r.v., we can also define a *probability mass function (p.m.f.)* $f_X : \Omega_X \mapsto \mathbb{R}$ with

$$f_X[x] = P(X = x).$$

- If we differentiate the c.d.f. w.r.t. x, we obtain the *probability distribution function (p.d.f.)* $f_X : \mathbb{R} \mapsto \mathbb{R}$, *i.e.*,

$$f_X(x) = \frac{dF(x)}{dx}, \qquad F_X(x) = \int_{-\infty}^{x} f_X(\xi)\, d\xi.$$

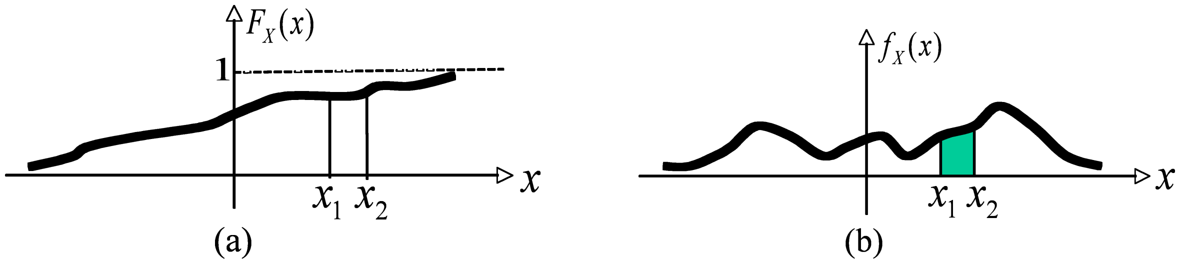
- For discrete r.v.s, there is a relationship between the p.m.f. and the p.d.f.:

$$f_X(x) = \sum_{\xi \in \Omega_X} f_X[\xi]\delta(x - \xi)$$

  where $\delta(x)$ is *Dirac delta function* for which $\delta(x) = 0$ when $x \neq 0$ and $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$.

- Properties of a p.d.f.:
  1. $f_X(x) \geqslant 0$,
  2. $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.
- It follows that

$$P(a \leqslant X \leqslant b) = F_X(b) - F_X(a) = \int_a^b f_X(x)\,dx.$$

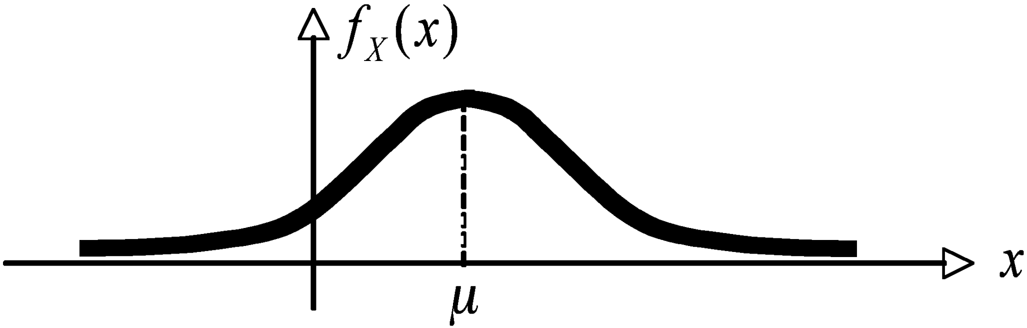


Important Discrete Distributions

Distribution	Definition	Written
Bernoulli	$f[0] = p, f[1] = q = 1 - p$	
Binomial	$f[k] = \binom{n}{k} p^k (1-p)^{n-k},$ $k = 0, \ldots, n$	$X \sim \mathrm{Bin}(n, p)$
Geometric	$f[k] = p(1-p)^{k-1}, k > 0$	$X \sim \mathrm{G}(p)$
Poisson	$f[k] = e^{-\lambda} \frac{\lambda^k}{k!}, k \geqslant 0$	$X \sim \mathrm{Po}(\lambda)$
Hyper-geometric	$f[k] = \frac{\binom{r}{k} \binom{N-r}{n-k}}{\binom{N}{n}},$ $\max\{0, r+n-N\} \leqslant k \leqslant \min\{n, r\}$	$X \sim \mathrm{Hyp}(n, r, N)$
Uniform	$f[k] = \frac{1}{b-a}, a \leqslant k \leqslant b$	$X \sim \mathrm{U}(a, b)$

Important Continuous Distributions

Distribution	Definition	Written
Uniform	$f(x) = \frac{1}{b-a}, a \leqslant x \leqslant b$	$X \sim \mathrm{U}(a, b)$
Exponential	$f(x) = \lambda e^{-\lambda x}, x \geqslant 0$	$X \sim \mathrm{Exp}(\lambda)$
Gamma	$f(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(x)}, x \geqslant 0,$ $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u}\, du, \alpha > 0$	$X \sim \mathrm{Gam}(\alpha, \lambda)$
Normal, Gaussian	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$	$X \sim \mathrm{N}(\mu, \sigma^2)$



Functions of a Random Variable

If X is a r.v. and $g : \mathbb{R} \mapsto \mathbb{R}$ is a function then $Y = g(X)$ is also a r.v.

- A simple but important case is the *affine transformation*

$$Y = aX + b.$$

- If X is a continuous r.v., the p.d.f. of Y can be expressed as

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

- If $g(x)$ is invertible, the p.d.f. of Y can be expressed more generally as

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|} \quad \text{where} \quad g'(x) = \frac{dg(x)}{dx} \quad \text{and} \quad x = g^{-1}(y).$$

- This expression is also applicable to discrete r.v.s, but it is necessary to remember that, when $x = g^{-1}(y)$,

$$\delta(x - x_0) = |g'(x)|\, \delta(y - y_0) \quad \text{where} \quad y_0 = g(x_0).$$

- If $g(x)$ is not invertible, we sum over the roots of $y = g(x)$.

Expectation

The *expectation* of r.v. X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

- More generally, the expectation of a function g of X is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

- For discrete r.v.s, this simplifies to

$$E[g(X)] = \sum_{x \in \Omega_X} g(x) f_X[x].$$

- An important property of expectation is *linearity*:

$$E[a_1 g_1(X) + \cdots + a_n g_n(X)] = a_1 E[g_1(X)] + \cdots + a_n E[g_n(X)].$$

- Some important expectations of a r.v.:
  - The *mean* $\mu_X = E[X]$.
  - The *variance* $\sigma_X^2 = E\left[(X - \mu_X)^2\right] = E[X^2] - \mu_X^2$.
  - The *standard deviation* σ_X.

Moments

The *n^{th} moment* of a r.v. X is defined as $E[X^n]$.

- It is common to define several other types of moments too:
  - the *n^{th} central moment* $E[(X - \mu_X)^n]$,
  - the *n^{th} absolute moment* $E[|X|^n]$,
  - the *n^{th} generalised moment* $E[(X - a)^n]$ for some a.
- The above admit some combinations, *e.g.*, the n^{th} central absolute moment is $E[|X - \mu_X|^n]$.

Inequalities

- The *Chebychev inequality* states that, for any $\epsilon > 0$,

$$P(|X - \mu_X| \geqslant \epsilon) \leqslant \frac{\sigma_X^2}{\epsilon^2}.$$

- Where $P(X < 0) = 0$, the *Markov inequality* states that, for any $x > 0$,

$$P(X \geqslant x) \leqslant \frac{\mu_X}{x}.$$

Characteristic & Moment Generating Functions

The *characteristic function* of a r.v. X is the Fourier transform of its p.d.f., *i.e.*,

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x}\, dx.$$

- Similarly, the Laplace transform of the p.d.f. yields the *moment generating function (m.g.f.)*

$$G_X(s) = \int_{-\infty}^{\infty} f_X(x) e^{sx}\, dx.$$

- From Fourier theory, we know that $\Phi_X(\omega) = G_X(j\omega)$.
- The usual Fourier & Laplace inversion formulae apply to obtain p.d.f.s from characteristic functions and m.g.f.s.
- The m.g.f. is so named because of the differentiation property:

$$G_X^{(n)}(s) = \frac{d^n G_X(s)}{ds^n} = E\left[X^n e^{sX}\right].$$

- Hence, the n^{th}moment can be obtain from the identity

$$E[X^n] = G_X^{(n)}(0).$$