Advanced Computational Techniques in Engineering

Lecture SP05: *Power Spectral Density*

This lecture:

1. Systems.
2. LTI Systems.
3. Power Spectral Density.
4. Filtering of Stochastic Processes.

Ref: PP pp. 393–420.

Systems

A *system* can be defined as an interconnection of operations that transforms input signals into output signals. In the simplest case — a *single-input, single-output (SISO)* system — we define an overall *operator* H that describes the transformation of the input into the output.

- In continuous time: $y(t) = H\{x(t)\}$.
- In discrete time: $y[n] = H\{x[n]\}$.

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

Continuous time

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

Discrete time

Memory, Linearity, Time Invariance

- A system has *memory* if its output does not depend on the current value of the input, but also on values at other times.
- A system is *time-invariant* if a time shift in the input causes only the corresponding time shift in the output. In the continuous-time case,

$$\text{if} \quad \xi(t) = x(t - t_0) \quad \text{then} \quad \eta(t) = H\{\xi(t)\} = y(t - t_0).$$

- A system is *linear* if it satisfies the so-called *principle of superposition*. Under this principle, the addition of scaled input signals produces the addition of the corresponding scaled output signals, *i.e.*,

$$\text{if} \quad \xi(t) = \lambda_1 x_1(t) + \lambda_2 x_2(t) \quad \text{then} \quad \eta(t) = \lambda_1 y_1(t) + \lambda_2 y_2(t).$$

- A very important and practical class of systems are *linear, time-invariant* (LTI) systems.
  - They possess several properties that make them attractive from a theoretical point of view.

LTI Systems

- One of the most important properties of an LTI system is that the system is characterised by its *impulse response*.
- Given the response of the system to an impulse, the response to any other signal can be computed in a straightforward manner.

The Impulse Response & Convolution

- If the input to a continuous-time LTI system is $\delta(t)$, label the output $h(t)$ — the *continuous-time impulse response*.

$$\delta(t) \longrightarrow \boxed{H} \longrightarrow h(t)$$

- The output is the *continuous-time linear convolution* of $x(t)$ and $h(t)$. We write

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau.$$

The Frequency Response

The *complex exponential* signal $e^{j2\pi ft}$ is an *eigenfunction* of an LTI system.

- That is, with $x(t) = e^{j2\pi ft}$,

  $$H\{x(t)\} = \lambda x(t)$$

  where λ is the corresponding *eigenvalue*.
- The eigenvalue in this case is called the *frequency response* and is a function of the frequency f, so we write $H(f)$.
- Where the Fourier transform exists, it turns out that

  $$h(t) \overset{FT}{\longleftrightarrow} H(f).$$

- The convolution property of the Fourier transform means that we can write an input-output equation in the frequency domain:

  $$Y(f) = H(f)X(f)$$

  where $X(f)$ and $Y(f)$ are the Fourier transforms of the input and output, respectively.

Power Spectral Density

We define the *power spectral density (PSD)* of a continuous-time stochastic process $X(t)$ as

$$S_X(f) = \lim_{T\to\infty} \frac{1}{T} E\Big[\,|X_T(f)|^2\Big]$$

where

$$X_T(f) = \int_{-T/2}^{T/2} X(t) e^{-j2\pi f t}\, dt.$$

The (Einstein-)Wiener-Khintchine Theorem

The *Wiener-Khintchine Theorem* states that, for WSS processes,

$$R_X(\tau) \overset{FT}{\longleftrightarrow} S_X(f).$$

- Hence, we may evaluate the PSD either directly or indirectly (calculate $R_X(\tau)$ first, then take FT).

Properties of the PSD

1. $S_X(f)$ is always real.
2. $S_X(f) \geqslant 0$.
3. When $X(t)$ is real, $S_X(f)$ is even.
4. $S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, d\tau$.

The Cross-Spectral Density

The *cross-spectral density* measures the frequency relationship between two stochastic processes.

- For jointly WSS stochastic processes $X(t)$ and $Y(t)$,

$$R_{XY}(\tau) \xleftrightarrow{FT} S_{XY}(f).$$

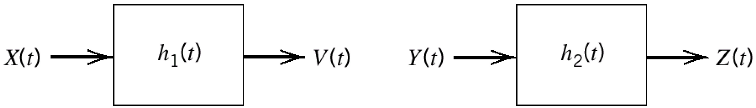
Filtering of Stochastic Processes

When the input to an LTI system is a stochastic process, $X(t)$, what can be said about the output, $Y(t)$?

- The usual input-output relationships hold for any realisation.
- When the input is WSS, the autocorrelations and PSDs are related:

$$R_Y(\tau) = h^*(-\tau) * h(\tau) * R_X(\tau),$$
$$S_Y(f) = |H(f)|^2 S_X(f).$$

Cross-Spectral Densities and Filtering



Consider a pair of LTI filters and processes as shown. We have

$$R_{VZ}(\tau) = h_1(\tau) * h_2^*(-\tau) * R_{XY}(\tau),$$
$$S_{VZ}(f) = H_1(f) H_2^*(f) S_{XY}(f).$$