Advanced Computational Techniques in Engineering

Lecture SP06: *Discrete-Time Stochastic Processes*

This lecture:

1. Correlation and Covariance.
2. Power Spectral Density.
3. Sampling.

Ref: PP pp. 420–426, 506–509.

Correlation and Covariance

Most of the definitions pertaining to continuous-time stochastic processes are adapted in an unsurprising way to discrete time.

- The *autocorrelation* and *autocovariance* of a (complex) process $X[n]$ are defined as

$$R_X[n_1, n_2] = E\big[X[n_1]X^*[n_2]\big],$$
$$C_X[n_1, n_2] = E\big[\{X[n_1] - \mu_X[n_1]\}\{X[n_2] - \mu_X[n_2]\}^*\big].$$

- If $X[n]$ is *SSS* then the joint distributions are invariant to a shift in the time origin.

- If $X[n]$ is *WSS* then

  1. $\mu_X[n] = \mu_X = \text{constant}$ and
  2. $R_X[n_1, n_2] = R_X[m]$ where $m = n_1 - n_2$ is the *lag*.

- In *discrete-time white noise*, the $X[n]$ are uncorrelated r.v.s.
  - We are usually concerned with discrete-time white noise which is zero-mean and WSS so that $R_X[m] = q\delta[m]$.

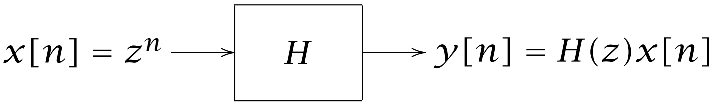
Discrete-Time Signals and Systems

A LTI discrete-time system, as for continuous-time, is characterised by its impulse response $h[n]$.

- Given the input $x[n]$, the output $y[n]$ can be determined by computing the *discrete-time linear convolution*

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m].$$

- It turns out that the eigenfunctions of LTI discrete-time systems have the form $x[n] = z^n$.
  - The associated eigenvalue is a function of z and this gives rise to the *transfer* (or *system*) *function* $H(z)$, *i.e.*,



The z-Transform and the Discrete-Time Fourier Transform

The impulse response and the transfer function are related through the *z-transform*:

$$h[n] \stackrel{z}{\longleftrightarrow} H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}.$$

- The convolution property of the z-transform means that we can write an input-output equation in the *z-domain*:

$$Y(z) = H(z)X(z).$$

- A special case of the eigenfunction z^n is when $z = e^{j\omega}$, in which case $z^n = e^{j\omega n}$ is a complex exponential signal.
- For $z = e^{j\omega}$, the transfer function is known as the *frequency response* $H(e^{j\omega})$.
  - The z-transform of a signal $x[n]$ for $z = e^{j\omega}$ is known as the *discrete-time Fourier transform (DTFT)* $X(e^{j\omega})$.

LTI Systems Governed by Difference Equations

A convenient way of modelling a very wide class of LTI systems is through a *difference equation* governing inputs and outputs:

$$\sum_{k=0}^{p} a[k]y[n-k] = \sum_{k=0}^{q} b[k]x[n-k].$$

- It turns out that the transfer function in this case is

$$H(z) = \frac{B(z)}{A(z)}.$$

- We are especially interested in the case where the input $X[n]$ is white noise.
- If $p = 0$ and $q > 0$ we say that the output $Y[n]$ is a *moving average* process, MA(q).
- If $q = 0$ and $p > 0$ then it is an *autoregressive* process, AR(p).
- If both $p > 0$ and $q > 0$ then it is an *autoregressive moving average* process, ARMA(p, q).

Power Spectral Density

We can define the PSD in terms of the DTFT so that

$$S_X(e^{j\omega}) = \lim_{N\to\infty} \frac{1}{2N+1} E\Big[\,|X_N(e^{j\omega})|^2\Big]$$

where $$X_N(e^{j\omega}) = \sum_{n=-N}^{N} X[n]e^{-j\omega n}.$$

- When WSS, the Wiener-Khintchine theorem holds:

$$R_X[m] \overset{DTFT}{\longleftrightarrow} S_X(e^{j\omega}).$$

- The *inverse discrete-time Fourier transform* can be used to obtain $R_X[m]$ from $S_X(e^{j\omega})$:

$$R_X[m] = \int_{-\pi}^{\pi} S_X(e^{j\omega})e^{j\omega m}\, d\omega.$$

- A property of the PSD is that $S_X(e^{j\omega}) \geqslant 0$ (except in pathological cases which we will not cover in this course).

Filtering Discrete-Time Stochastic Processes

The continuous-time results transfer to discrete-time in an obvious way.

- When the input to a LTI system is WSS, the autocorrelations and PSDs of input $X[n]$ and output $Y[n]$ are related:

$$R_Y[m] = h[m] * h^*[-m] * R_X[m],$$
$$S_Y(e^{j\omega}) = |H(e^{j\omega})|^2 S_X(e^{j\omega}).$$

Sampling

In digital signal processing, it is common to sample an analogue signal $x_c(t)$ to produce a digital or discrete-time signal $x[n]$ so that

$$x[n] = x_c(nT_s)$$

where T_s is the *sampling period* and $f_s = 1/T_s$ is the *sampling rate*.

- So long as $x_c(t)$ is properly *bandlimited*, *i.e.*, $X_c(f) = 0$ when $|f| \geqslant \frac{1}{2} f_s$, we can reconstruct $x_c(t)$ perfectly from $x[n]$.
- This is the basis for the operation and widespread use of *digital-to-analogue (D/A)* and *analogue-to-digital (A/D) converters*.
- What happens for stochastic processes?
- For WSS processes, $\mu_X = \mu_{X_c}$ and the autocorrelation is sampled:

$$R_X[m] = R_{X_c}(mT_s).$$

- For the *normalised frequency* $-\frac{1}{2} \leqslant \alpha \leqslant \frac{1}{2}$, we also have for band-limited processes that

$$S_X(e^{j2\pi\alpha}) = S_{X_c}(\alpha f_s).$$