On Parameter Values for a Mahalanobis Distance Fuzzy Classifier

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Abstract

The fuzzy c-means clustering algorithm (and a supervised classifier based on it) requires the *a priori* selection of a weighting parameter called the fuzzy exponent (denoted m). Guidance in the existing literature on an appropriate value of m is not definitive. This paper determines suitable values of m by using the criterion that fuzzy set memberships reflect class proportions in the mixed pixels of a remotely sensed image.

Keywords: Pattern Classification, Fuzzy Sets, Mahalanobis Distance.

1 Introduction

The unsupervised fuzzy c-means (FCM) clustering algorithm [1] has found widespread use in pattern recognition applications. A supervised variant of this algorithm was proposed in Key *et al.* [10], and used in Foody [7]. Our implementation is as follows: training sets representative of the desired classes are selected. Training class means and covariance matrices are computed, and used to determine the (squared) Mahalanobis distance between each pixel and the mean of each class, $d_{ik}^2 = (x_k - m_i)^t \mathbf{\Sigma}_i^{-1}(x_k - m_i)$ where d_{ik}^2 represents the (squared) Mahalanobis distance between pixel x_k and the mean of class i, m_i; superscript t denotes (vector) transpose; and $\mathbf{\Sigma}_i^{-1}$ is the inverse of the covariance matrix of the ith class (for singular matrices, the identity matrix is chosen). Fuzzy class memberships are calculated according to, $\mu_{ik} = \frac{(\frac{1}{d_{ik}^2})^m}{\sum_{j=1}^{c}(\frac{1}{d_{jk}^2})^m}$ $i = 1,\ldots,c$ and $k = 1,\ldots,n$, where μ_{ik} is the fuzzy class membership of pixel x_k in class i, c is the number of classes, n is the number of pixels, and m is a parameter used to control the "degree of fuzziness" of the class membership. m is related to the fuzzy exponent in FCM (denoted as q) by, $m = \frac{1}{q-1}$. This substitution has been effected for subsequent notational ease.

With both the unsupervised FCM clustering algorithm and the supervised Mahalanobis distance fuzzy classifier, *a priori* choices about the number of classes c and the fuzzy exponent m are required. When using the supervised Mahalanobis distance fuzzy classifier for a specific application and set of data, the number of classes c is determined by the application, but the selection the value of q needs to addressed. Bezdek *et al.* [1] suggest that, for the unsupervised FCM clustering algorithm, q should be in the range 1 to 30, with the range 1.5 to 3 seeming "to give good results" — these values for q correspond to values of m in the range positive ∞ to 0.03, with "good" results in the range 2.0 to 0.5. It was noted, however, that there was no strong theoretical justification or empirical evidence for these choices. Cannon *et al.* [4] suggested $1.1 \leq q \leq 5$ were "typically reported as the most useful range of values". McBratney and Moore [11] investigated the choice of q and found that a value of ≈ 2 (corresponding to $m \approx 1$) was optimal whereas Choe and Jordan [5] found that the unsupervised FCM algorithm was insensitive to the value chosen for q in the range 1.1 to 30. Cannon and Jacobs [3] found that, for image applications, $1.1 \leq q \leq 2.5$ "proved adequate for all practical purposes".

2 Mixed Pixels

The adoption of fuzzy classification approaches has been motivated by the presence of the mixed pixel problem. This occurs when there are a number of classes contributing to the observed spectral response of a pixel. Some earlier work on the interpretation of fuzzy set memberships of pixels in remotely sensed images as mixed pixel class proportions has produced good results. Fisher and Pathirana [6], Wang [13], Foody [7, 8], Foody and Cox [9] and Maselli *et al.* [12] have all suggested, with varying degrees of empirical support, that there is a strong relationship between fuzzy memberships and proportions of ground cover.

The interpretation of fuzzy class memberships as proportional representation offers an approach to determining a suitable value for the fuzzy exponent m in the supervised Mahalanobis distance fuzzy classifier used in this study (and by extension, the unsupervised FCM clustering algorithm).

3 Empirical investigation of m

Maselli *et al.* [12] notes that "quantitative investigations on the actual relationships existing between fuzzy membership grades and cover proportions would be valuable". We approach this issue in the opposite direction. The criterion that the computed fuzzy memberships of a pixel should reflect the true class proportions of that pixel is used in determining a suitable value for the fuzzy exponent m in the supervised Mahalanobis distance fuzzy classifier. This includes the special case where the pixel actually comprises a single class.

A study area located towards the western end of Kangaroo Island, South Australia, was selected. The datasets were a 1993 Landsat 5 TM (partial) image (1024 × 1024 pixels, bands 2, 3, 4 and 5), and a corresponding digital ground truth dataset. The classes of primary interest were forest, sea, pasture and pine.

Using class knowledge from the ground truth data, we calculate expected image values for each class. Using these, and the training class means and covariance matrices, we compute the expected Mahalanobis distances of a typical member of each class from each of the class means (shown in Table 1).

Table 1: Expected Mahalanobis distances (mahdist) of a typical member of each class from each class mean.

mean mahdist, given actual class				
	actual class			
class mahdist	forest	sea	pasture	pine
forest	6.2	107.2	100.9	14.6
sea	5769.0	5.1	24094.0	5802.0
pasture	304.2	937.3	16.0	476.8
pine	403.6	417.3	3406.0	3.8

We denote the forest (squared) Mahalanobis distance of a pixel by m_f, the sea (squared) Mahalanobis distance of a pixel by m_s, the pasture (squared) Mahalanobis distance of a pixel by m_p, and the pine (squared) Mahalanobis distance of a pixel by m_π. The fuzzy class membership of a particular pixel is then given by,

$$y = \frac{\left(\frac{1}{m_f}\right)^x}{\left(\frac{1}{m_f}\right)^x + \left(\frac{1}{m_s}\right)^x + \left(\frac{1}{m_p}\right)^x + \left(\frac{1}{m_\pi}\right)^x}$$

where y represents the fuzzy class membership, and x represents the fuzzy exponent m. This temporary change in notation is motivated by a view of the fuzzy exponent m as a variable, rather than a fixed, but, as yet, undetermined value.

3.1 Pure pixels

Consider the specific case where the pixel is actually a member of class forest. Such a pixel should exhibit a high degree of membership (say, > 0.9) in the class forest, and a low degree of membership (< 0.1) in all other classes.

Selecting the appropriate values from Table 1, we have the membership of such a pixel in the class forest given by,

$$y = \frac{\left(\frac{1}{6.2}\right)^x}{\left(\frac{1}{6.2}\right)^x + \left(\frac{1}{5769}\right)^x + \left(\frac{1}{304.2}\right)^x + \left(\frac{1}{403.6}\right)^x}$$

The fuzzy class membership is plotted as a function of the exponent x in Fig. 1.

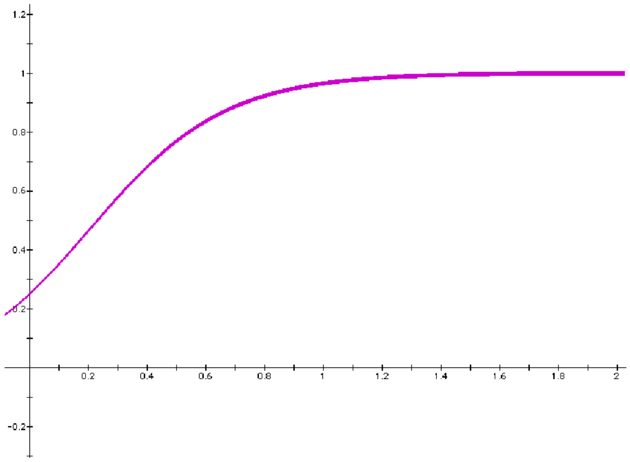

Figure 1: Membership of a pure pixel of class forest, in the class forest, as a function of the fuzzy exponent

The desired membership in the class forest of > 0.9 occurs at $x \geq 0.8$. Because of the unity sum constraint on the class memberships of a given pixel, the sum of the other memberships in the other classes will be one minus the membership of forest.

Similar analysis of pure pixels in each of the other classes reveals a requirement for x to be at least 0.9 (sea), 1.3 (pasture) and 1.7 (pine).

As these are the minimum values of the fuzzy exponent required to achieve the desired fuzzy class memberships, a value of m of approximately 2 is shown, with these data (and in the case of pure pixels), to give satisfactory results. This value of m corresponds to a value of $q = 1.5$.

3.2 Mixed pixels

Consider now the case of a pixel comprising a mixture of two classes. Using a linear mixing model, a synthetic pixel of arbitrary proportions can be constructed. Table 2 shows the expected Mahalanobis distances from all the class means of a synthetic pixel comprising equal two class mixtures of forest/pasture, forest/pine and pasture/pine.

Using these data, the fuzzy set membership in the class forest of a pixel comprising 50% forest and 50% pasture is given by,

$$y = \frac{\left(\frac{1}{26.9}\right)^x}{\left(\frac{1}{26.9}\right)^x + \left(\frac{1}{12435.1}\right)^x + \left(\frac{1}{83.9}\right)^x + \left(\frac{1}{1380.7}\right)^x}$$

Table 2: Expected Mahalanobis distances (mahdist) for a two class mixed synthetic pixel (equal proportions) based on TM93 image data

	mixed pixel comprising equal proportions of		
mahdist class	forest/pasture	forest/pine	pasture/pine
forest	26.9	3.2	30.6
sea	12435.1	4826.3	11190.1
pasture	83.9	398.6	119.8
pine	1380.7	49.3	935.9

The fuzzy class membership is plotted as a function of the exponent x in Fig. 2.

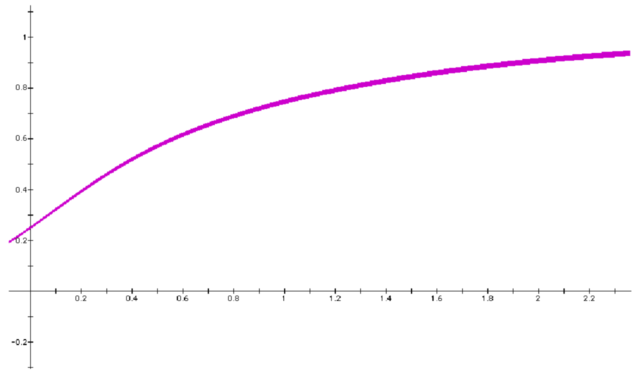

Figure 2: Membership of a mixed pixel (forest/pasture) in the class forest as a function of the fuzzy exponent

The desired membership in the class forest is 0.5 (reflecting its true proportion in the class forest). Tshis is achieved at a value of x of about 0.4. We can see from this graph that if we selected an x value of 2 as suggested by our analysis of the requirement for pure pixels, this mixed pixel would have a forest membership of approximately 0.9. The membership of the pixel in the class pasture is given by,

$$y = \frac{\left(\frac{1}{83.9}\right)^x}{\left(\frac{1}{26.9}\right)^x + \left(\frac{1}{12435.1}\right)^x + \left(\frac{1}{83.9}\right)^x + \left(\frac{1}{1380.7}\right)^x}$$

This is shown in Fig. 3. This function does not achieve the desired value of 0.5: it achieves its maximum value (approximately 0.3) in the range $0.3 < x < 0.5$. A similar consideration of a mixed pixel of equal proportions of forest and pine reveals a requirement for $m = 0.25$ (forest) and $x = 0.15$ (pine).

The case of the pixel comprising mixed pasture and pine is disturbing: its fuzzy set memberships

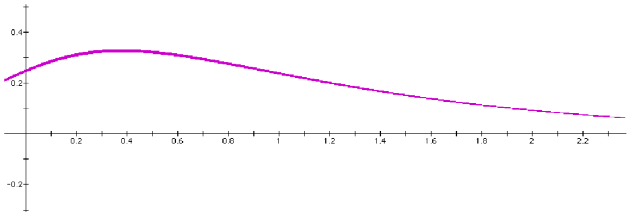

Figure 3: Membership of a mixed pixel (forest/pasture) in the class pasture as a function of the fuzzy exponent

reveal a high value (monotonically increasing with the exponent x, and approaching 1 by $x = 2.5$) in the fuzzy set forest, even though forest is not one of the constituent classes! This shows that a fuzzy classification approach is not quarantined from the problem observed in Campbell [2], that a conventional classification will force the allocation of a mixed pixel to a single class, with this class not necessarily even being present in the pixel.

4 Conclusion

Empirical investigation reveals conflicting requirements for the selection of the fuzzy exponent m in a supervised fuzzy classifier based on the ratios of the reciprocals of the class Mahalanobis distances. For pure pixels we want the value of m to be about 2 (or higher), giving a high membership in the pure class that the pixel is actually a member of. For mixed pixels, we want it to be less than 0.5, thereby distributing the memberships between the two classes that actually comprise the pixel.

These values of m correspond to values of q in the fuzzy c-means clustering algorithm of $1.5 \leq q \leq 3$. This supports the findings of earlier studies and, through the nature of this empirical work, offers a physical interpretation of these values.

References

[1] Bezdek J.C., R. Ehrlich and W. Full (1984). FCM: The Fuzzy c-Means Clustering Algorithm. *Computers and Geoscience*, 10, pp. 191 - 203.

[2] Campbell J.B. (1987). *Introduction to Remote Sensing*. Guilford Press, New York.

[3] Cannon R.L. and C. Jacobs (1984). *Multispectral Pixel Classification with Fuzzy Objective Functions*. Center for Automation Research, Technical Report CAR-TR-51, University of Maryland, College Park.

[4] Cannon R.L., J.V. Dave and J.C. Bezdek (1986). *Efficient Implementation of the Fuzzy c-means Clustering Algorithm*. IEEE Transactions on Pattern Analysis and Machine Intelligence, 8, pp. 248-255.

[5] Choe H. and J.B Jordan (1992). On the optimal choice of parameters in a fuzzy c-means algorithm, Proceedings of IEEE International Conference on Fuzzy Systems, pp. 349-354.

[6] Fisher P.F. and S. Pathirana (1990). The Evaluation of Fuzzy Membership of Land Cover Classes in the Suburban Zone. *Remote Sensing of Environment*, 34, pp. 121-132.

[7] Foody G.M. (1992). A Fuzzy Sets Approach to the Representation of Vegetation Continua from Remotely Sensed Data: An Example from Lowland Heath. *Photogrammetric Engineering and Remote Sensing*, 58, pp. 221 - 225.

[8] Foody G.M. (1994). Ordinal-Level Classification of Sub-Pixel Tropical Forest Cover. *Photogrammetric Engineering and Remote Sensing*, 60, pp. 61-65.

[9] Foody G.M. and D.P. Cox (1994). Sub-Pixel Land Cover Composition Estimation Using a Linear Mixture Model and Fuzzy Set Membership. *International Journal of Remote Sensing*, 15, pp. 619-631.

[10] Key J.R., J.A. Maslanik and R.G. Barry (1989). Cloud Classification From Satellite Data Using a Fuzzy Sets Algorithm: A Polar Example. *International Journal of Remote Sensing*, 10, pp. 1823-1842.

[11] McBratney A.B. and A.W. Moore (1985). Application of Fuzzy Sets to Climatic Classification. *Agricultural and Forest Meteorology*, 35, pp. 165-185.

[12] Maselli F., A. Rodolfi and C. Conese (1996). Fuzzy Classification of Spatially Degraded Thematic Mapper Data for Estimation of Sub-Pixel Components. *International Journal of Remote Sensing*, 17, pp. 537-551.

[13] Wang F. (1990). Fuzzy Supervised Classification of Remotely Sensed Images. *IEEE Transactions on Geoscience and Remote Sensing*, 28, pp. 194-201.