Chapter 5

Z Semantics for OWL: Soundness Proof Using Institution Morphisms

As mentioned in the previous chapter, the validity of the combined approach depends on the correctness of the Z/Alloy semantics of the ontology languages. Since the Z and Alloy semantics are very similar to each other, we will focus on one of these, i.e., Z.

As OWL has become the W3C recommendation as the ontology language for the Semantic Web, it is necessary to extend the Z/Alloy support from DAML+OIL to OWL. In the OWL species, OWL DL retains decidability and is more expressive than OWL Lite, we have constructed the Z semantics for OWL DL, which can be found in Appendix C.

Institutions and institution morphisms are a powerful tool to abstract and reason about software systems without any assumption about the underlying logical systems. They make a perfect candidate to reason about the relationship between OWL and Z as they are based on description logics and first-order predicate logic respectively.

In this chapter, we use institutions to investigate the Z semantics of OWL DL. It is proved at the end of the chapter, by making use of the Z semantics for OWL, that there exists a comorphism between OWL DL and Z, meaning that Z is more expressive than OWL DL.

This chapter is divided into four parts. In Sections 5.1 and 5.2, we construct the institutions for OWL and Z, respectively. Section 5.3 is devoted to relating the two institutions. Finally, Section 5.4 concludes the chapter.

5.1 The OWL Institution $\mathfrak{O}$

In this section we briefly introduce the definition of the logic underlying the Web Ontology Language OWL DL. We note that in OWL DL there is mutual disjointness between classes, properties, and individuals.

We suppose that all the OWL specifications share the same data types. Therefore we consider given a set $\mathbb{D}$ of *data type names*, a set $\mathcal{V}$ of *data values*, and a function $[\![\_]\!]$ which associates a subset $[\![D]\!] \subseteq \mathcal{V}$ with each data type name D. The set of *data expressions* is defined as follows:

$$\mathcal{D} ::= D \mid \{v_1, \ldots, v_n\}$$

where D ranges over data type names and v_i ranges over data values. We extend the definition of $[\![\_]\!]$ by setting $[\![\{v_1, \ldots, v_n\}]\!] = \{v_1, \ldots, v_n\}$. In OWL definition [80] a data type D is characterized by a lexical space, $L(D)$, a value space, $V(D)$, and a mapping $L2V(D) : L(D) \to V(D)$. We represent a data type in a more abstract way by forgetting the lexical space. $V(D)$ is denoted here by $[\![D]\!]$. For instance, $(\mathbb{D}, [\![\_]\!])$ might be the set of the XML data types and/or the set of the OWL built-in types. We separate the data world from the world over which we define ontologies.

A first reason for this separation is that the specification of the data types is quite different from that of ontologies. Another reason is that we get more flexibility in relating web ontologies with various formalisms. For instance, we may use directly the built-in implementations of the data types in these formalisms and focus only on the translation of the taxonomy and its sentences.

An *OWL signature* consists of a quadruple $\mathcal{O} = (\mathbb{C}, \mathbb{R}, \mathbb{U}, \mathbb{I})$, where $\mathbb{C}$ is the set of the *concept (class) names*, $\mathbb{R}$ is the set of the *individual-valued property names*, $\mathbb{U}$ is the set of the *data-valued property names*, and $\mathbb{I}$ is the set of *individual names*. We suppose that $\mathbb{D}$, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{U}$, and $\mathbb{I}$ are pairwise disjoint. We denote by $\mathcal{N}(\mathcal{O})$ the set $\mathbb{C} \cup \mathbb{R} \cup \mathbb{U} \cup \mathbb{I}$. An *OWL signature morphism* $\phi : (\mathbb{C}, \mathbb{R}, \mathbb{U}, \mathbb{I}) \to (\mathbb{C}', \mathbb{R}', \mathbb{U}', \mathbb{I}')$ consists of a quadruple of functions $\phi = (\phi_{co}, \phi_{op}, \phi_{dp}, \phi_{in})$ where $\phi_{co} : \mathbb{C} \to \mathbb{C}'$, $\phi_{op} : \mathbb{R} \to \mathbb{R}'$, $\phi_{dp} : \mathbb{U} \to \mathbb{U}'$, and $\phi_{in} : \mathbb{I} \to \mathbb{I}'$. Sometimes we see ϕ as a function $\phi : \mathcal{N}(\mathcal{O}) \to \mathcal{N}(\mathcal{O}')$. We denote by $\mathsf{Sign}(\mathfrak{O})$ the category of the OWL signatures. Given an OWL signature $\mathcal{O} = (\mathbb{C}, \mathbb{R}, \mathbb{U}, \mathbb{I})$, an $\mathcal{O}$-*structure (model)* is a tuple $A = (\Delta_A, [\![\_]\!]_A, Res_A, res_A)$ consisting of a set of *resources* Res_A, a subset $\Delta_A \subseteq Res_A$ called *domain*, a function $res_A : \mathcal{N}(\mathcal{O}) \to Res_A$ associating a resource to each name in $\mathcal{O}$, and an interpretation function $[\![\_]\!]_A : \mathbb{C} \cup \mathbb{R} \cup \mathbb{U} \to \mathcal{P}(Res) \cup P(Res) \times P(Res)$ such that the following conditions hold:

$$
\begin{aligned}
&\mathcal{V} \subseteq Res_A,\\
&\Delta_A \cap \mathcal{V} = \emptyset,\\
&[\![C]\!]_A \subseteq \Delta_A \text{ for each } C \in \mathbb{C},\\
&[\![R]\!]_A \subseteq \Delta_A \times \Delta_A \text{ for each } R \in \mathbb{R},\\
&[\![U]\!]_A \subseteq \Delta_A \times \mathcal{V} \text{ for each } U \in \mathbb{U},\\
&res_A(o) \in \Delta_A \text{ for each } o \in \mathbb{I}.
\end{aligned}
$$

In order to have a uniform notation, we often write $[\![o]\!]_A$ for $res_A(o)$.

The definition above corresponds to that of abstract interpretation of an OWL vocabulary given by the direct model-theoretic semantics [80]. In particular we have $\Delta_A = O$, $[\![\_]\!]_A\,|_{\mathbb{C}} = EC$, $[\![\_]\!]_A\,|_{\mathbb{R}\cup\mathbb{U}} = ER$, and $res_A = S$. Here $[\![\_]\!]_A\,|_X$ denotes the restriction of the function $[\![\_]\!]_A$ to the subset X.

Given two $\mathcal{O}$-structures $A = (\Delta_A, [\![\_]\!]_A, Res_A, res_A)$ and $A' = (\Delta_{A'}, [\![\_]\!]_{A'}, Res_{A'}, res_{A'})$, an $\mathcal{O}$-*homomorphism* $h : A \to A'$ is a function $h : Res_A \to Res_{A'}$ such that:

1. $h(\Delta_A) = \Delta_{A'}$;
2. $res_{A'} = res_A;\ h$;
3. for each $C \in \mathbb{C}$ and $x \in \Delta_A$, $x \in [\![C]\!]_A$ iff $h(x) \in [\![C]\!]_{A'}$;
4. for each $R \in \mathbb{R}$ and $x, y \in \Delta_A$, $(x, y) \in [\![R]\!]_A$ iff $(h(x), h(y)) \in [\![R]\!]_{A'}$;
5. for each $U \in \mathbb{U}$, $x \in \Delta_A$, and $v \in \mathcal{V}$, $(x, v) \in [\![U]\!]_A$ iff $(h(x), v) \in [\![U]\!]_{A'}$.

Let $\mathsf{Mod}(\mathfrak{O})(\mathcal{O})$ denote the category of the $\mathcal{O}$-models. If $\phi : \mathcal{O} \to \mathcal{O}'$ is an OWL signature morphism and $A' = (\Delta_{A'}, [\![\_]\!]_{A'}, Res_{A'}, res_{A'})$ an $\mathcal{O}'$-structure, then the ϕ-reduct $A'\!\restriction_\phi$ is the $\mathcal{O}$-structure $A = (\Delta_A, [\![\_]\!]_A, Res_A, res_A)$, where $Res_{A\restriction\phi} = Res_{A'}$, $\Delta_{A\restriction\phi} = \Delta_{A'}$ and $res_A(N) = res_{A'}(\phi(N))$ for each name $N \in \mathcal{N}(\mathcal{O})$, and the interpretation function $[\![\_]\!]_A$ is defined as follows:

$[\![C]\!]_A = [\![\phi_{co}(C)]\!]_{A'}$ for each $C \in \mathbb{C}$;
$[\![R]\!]_A = [\![\phi_{op}(R)]\!]_{A'}$ for each $R \in \mathbb{R}$;
$[\![U]\!]_A = [\![\phi_{dp}(U)]\!]_{A'}$ for each $U \in \mathbb{U}$.

If $h' : A' \to A''$ is an $\mathcal{O}'$-homomorphism, then the reduct along ϕ of h' is the $\mathcal{O}$-homomorphism $h'\!\restriction_\phi : A'\!\restriction_\phi \to A''\!\restriction_\phi$ given by $h'\!\restriction_\phi(x') = h'(x')$. It is a matter of routine to check that $h'\!\restriction_\phi$ is indeed an $\mathcal{O}$-homomorphism. We may now consider the functor $\mathsf{Mod}(\mathfrak{O}) : \mathsf{Sign}(\mathfrak{O})^{\mathrm{op}} \to \mathsf{Cat}$ mapping each OWL signature $\mathcal{O}$ to the category of its models $\mathsf{Mod}(\mathfrak{O})(\mathcal{O})$ and each OWL signature morphism $h : \mathcal{O} \to \mathcal{O}'$ to the forgetful functor $\mathsf{Mod}(\mathfrak{O})(\phi^{\mathrm{op}}) : \mathsf{Mod}(\mathfrak{O})(\mathcal{O}') \to \mathsf{Mod}(\mathfrak{O})(\mathcal{O})$ defined by $\mathsf{Mod}(\mathfrak{O})(\phi^{\mathrm{op}})(h') = h'\!\upharpoonright_{\phi}$.

The set of the $\mathcal{O}$-*expressions* is defined by:

$$\begin{aligned}
\mathcal{C} ::= & \bot \mid \top \mid C \mid \mathcal{C} \sqcap \mathcal{C} \mid \mathcal{C} \sqcup \mathcal{C} \mid \neg\, \mathcal{C} \\
& \mid \forall\, \mathcal{R}.\mathcal{C} \mid \exists\, \mathcal{R}.\mathcal{C} \mid \leqslant n\, \mathcal{R} \mid \geqslant n\, \mathcal{R} \mid R : o \\
& \mid \forall\, U.\mathcal{D} \mid \exists\, U.\mathcal{D} \mid \leqslant n\; U \mid \geqslant n\; U \mid U : v \\
& \mid \{o_1, \ldots, o_n\} \\
\mathcal{R} ::= & R \mid \mathsf{Inv}(R)
\end{aligned}$$

where C ranges over concepts names, R ranges over individual-valued properties names, U over data-valued properties, v over $\mathcal{V}$, and o_i over individuals names.

The set of OWL $\mathcal{O}$-*sentences* is defined by:

$$\begin{aligned}
\mathcal{F} ::= & \mathcal{C} \sqsubseteq \mathcal{C} \mid \mathcal{C} \equiv \mathcal{C} \mid \mathsf{Disjoint}(\mathcal{C}, \ldots, \mathcal{C}) \\
& \mid \mathsf{Tr}(R) \mid \mathcal{R} \sqsubseteq \mathcal{R} \mid \mathcal{R} \equiv \mathcal{R} \\
& \mid U \sqsubseteq U \mid U \equiv U \\
& \mid o : \mathcal{C} \mid (o, o') : \mathcal{R} \mid (o, v) : U \mid o \equiv o' \mid o \not\equiv o'
\end{aligned}$$

where n ranges over natural numbers, o and o' over individuals names, and v over data values. We denote by $\mathsf{sen}(\mathfrak{O})(\mathcal{O})$ the set of the OWL $\mathcal{O}$-sentences. If $\phi : \mathcal{O} \to \mathcal{O}'$ is an OWL signature morphism, then $\mathsf{sen}(\mathfrak{O})(\phi) : \mathsf{sen}(\mathfrak{O})(\mathcal{O}) \to \mathsf{sen}(\mathfrak{O})(\mathcal{O}')$ is the function translating the OWL $\mathcal{O}$-sentences in OWL $\mathcal{O}'$-sentences in the standard way; for instance,

$$\mathsf{sen}(\mathfrak{O})(\phi)(\forall\, R.C \sqcap C') = \forall\, \phi_{op}(R).\phi_{co}(C) \sqcap \phi_{co}(C').$$

We have now defined the functor

$$\mathsf{sen}(\mathfrak{O}) : \mathsf{Sign}(\mathfrak{O}) \to \mathsf{Set}.$$

Example 1 *Here is a very simple example of OWL specification:*

$$
\begin{aligned}
\mathbb{C} &= \{Author,\ FamousAuthor,\ Paper\},\\
\mathbb{R} &= \{writtenBy,\ citedBy\},\\
\mathbb{U} &= \{noOfPages\},\\
\mathbb{I} &= \{Kleene, Mathematical\ Logic\},\\
\\
\mathcal{F} &= \{FamousAuthor \sqsubseteq Author,\\
&\quad Paper \sqsubseteq\ \geqslant 1\ writtenBy,\\
&\quad \top \sqsubseteq \forall\, writtenBy.Author,\\
&\quad Paper \sqsubseteq\ \geqslant 1\ citedBy,\\
&\quad \top \sqsubseteq \forall\, citedBy.Author,\\
&\quad \geqslant 1\ noOfPages \sqsubseteq Paper,\\
&\quad \top \sqsubseteq \forall\, noOfPages.integer,\\
&\quad (Mathematical\ Logic, Kleene) : writtenBy,\\
&\quad Kleene : FamousAuthor\}
\end{aligned}
$$

The first sentence asserts that any famous author is an author. The second one asserts that `Paper` *is included in the domain of the individual-valued property* `writtenBy`*. The third sentence asserts that the range (codomain) of* `writtenBy` *is included in* `Author`*. We show the validity of these two assertions later when we give the semantics for expressions and sentences. The next four sentences are similar to the second and the third, respectively. The last two sentences are self-explanatory.*

The semantics of the $\mathcal{O}$-expressions is given by:

$$
\begin{aligned}
&[\![\bot]\!]_A = \emptyset,\\
&[\![\top]\!]_A = \Delta_A,\\
&[\![\mathsf{Inv}(R)]\!]_A = \{(y,x) \mid (x,y) \in [\![R]\!]_A\},\\
&[\![\mathcal{C} \sqcap \mathcal{C}']\!]_A = [\![\mathcal{C}]\!]_A \cap [\![\mathcal{C}']\!]_A,\\
&[\![\mathcal{C} \sqcup \mathcal{C}']\!]_A = [\![\mathcal{C}]\!]_A \cup [\![\mathcal{C}']\!]_A,\\
&[\![\neg\, \mathcal{C}]\!]_A = \Delta_A \setminus [\![\mathcal{C}]\!]_A,
\end{aligned}
$$

$$
\begin{aligned}
&[\![\forall\,\mathcal{R}.\mathcal{C}]\!]_A = \{x \mid (\forall\, y)(x,y) \in [\![\mathcal{R}]\!]_A \Rightarrow y \in [\![\mathcal{C}]\!]_A\},\\
&[\![\exists\,\mathcal{R}.\mathcal{C}]\!]_A = \{x \mid (\exists\, y)(x,y) \in [\![\mathcal{R}]\!]_A \wedge y \in [\![\mathcal{C}]\!]_A\},\\
&[\![\leqslant n\,\mathcal{R}]\!]_A = \{x \mid \#(\{y \mid (x,y) \in [\![\mathcal{R}]\!]_A\}) \leqslant n\},\\
&[\![\geqslant n\,\mathcal{R}]\!]_A = \{x \mid \#(\{y \mid (x,y) \in [\![\mathcal{R}]\!]_A\}) \geqslant n\},\\
&[\![R : o]\!]_A = \{x \mid (x, [\![o]\!]_A) \in [\![R]\!]_A\},\\
&[\![\forall\, U.\mathcal{D}]\!]_A = \{x \mid (\forall\, v)(x,v) \in [\![U]\!]_A \Rightarrow v \in [\![\mathcal{D}]\!]\},\\
&[\![\exists\, U.\mathcal{D}]\!]_A = \{x \mid (\exists\, v)(x,v) \in [\![U]\!]_A \wedge v \in [\![\mathcal{D}]\!]\},\\
&[\![\leqslant n\; U]\!]_A = \{x \mid \#(\{v \mid (x,v) \in [\![U]\!]_A\}) \leqslant n\},\\
&[\![\geqslant n\; U]\!]_A = \{x \mid \#(\{v \mid (x,v) \in [\![U]\!]_A\}) \geqslant n\},\\
&[\![U : v]\!]_A = \{x \mid (x,v) \in [\![U]\!]_A\},\\
&[\![\{o_1,\ldots,o_n\}]\!]_A = \{res_A(o_1),\ldots,res_A(o_n)\}.
\end{aligned}
$$

The satisfaction relation between $\mathcal{O}$-structures and $\mathcal{O}$-sentences is defined as follows:

$$
\begin{aligned}
&A \models_{\mathcal{O}} \mathcal{C} \sqsubseteq \mathcal{C}' \text{ iff } [\![\mathcal{C}]\!]_A \subseteq [\![\mathcal{C}']\!]_A,\\
&A \models_{\mathcal{O}} \mathcal{C} \equiv \mathcal{C}' \text{ iff } [\![\mathcal{C}]\!]_A = [\![\mathcal{C}']\!]_A,\\
&A \models_{\mathcal{O}} \mathsf{Disjoint}(\mathcal{C}_1,\ldots,\mathcal{C}_n), \text{ iff } [\![\mathcal{C}_i]\!]_A \cap [\![\mathcal{C}_j]\!]_A = \emptyset \text{ for all } i \neq j,\\
&A \models_{\mathcal{O}} \mathsf{Tr}(\mathcal{R}) \text{ iff } [\![\mathcal{R}]\!]_A \text{ is transitive},\\
&A \models_{\mathcal{O}} \mathcal{R} \sqsubseteq \mathcal{R}' \text{ iff } [\![\mathcal{R}]\!]_A \subseteq [\![\mathcal{R}']\!]_A,\\
&A \models_{\mathcal{O}} \mathcal{R} \equiv \mathcal{R}' \text{ iff } [\![\mathcal{R}]\!]_A = [\![\mathcal{R}']\!]_A,\\
&A \models_{\mathcal{O}} U \sqsubseteq U' \text{ iff } [\![U]\!]_A \subseteq [\![U']\!]_A,\\
&A \models_{\mathcal{O}} U \equiv U' \text{ iff } [\![U]\!]_A = [\![U']\!]_A,\\
&A \models_{\mathcal{O}} o : \mathcal{C} \text{ iff } [\![o]\!]_A \in [\![\mathcal{C}]\!]_A,\\
&A \models_{\mathcal{O}} (o,o') : \mathcal{R} \text{ iff } ([\![o]\!]_A, [\![o']\!]_A) \in [\![\mathcal{R}]\!]_A,\\
&A \models_{\mathcal{O}} (o,v) : U \text{ iff } ([\![o]\!]_A, v) \in [\![U]\!]_A,\\
&A \models_{\mathcal{O}} o \equiv o' \text{ iff } [\![o]\!]_A = [\![o']\!]_A,\\
&A \models_{\mathcal{O}} o \not\equiv o' \text{ iff } [\![o]\!]_A \neq [\![o']\!]_A.
\end{aligned}
$$

Example 2 *We have:*

$$
\begin{aligned}
&A \models \quad \mathit{Paper} \sqsubseteq \quad \geqslant 1\, \mathit{writtenBy} \quad \mathit{iff}\\
&\quad [\![\mathit{Paper}]\!]_A \subseteq [\![\, \geqslant 1\, \mathit{writtenBy}]\!]_A \quad \mathit{iff}\\
&\quad [\![\mathit{Paper}]\!]_A \subseteq \{x \mid \#(\{y \mid (x,y) \in [\![\mathit{writtenBy}]\!]_A\}) \geqslant 1\}\\
&\quad \mathit{iff}\\
&\quad [\![\mathit{Paper}]\!]_A \subseteq \{x \mid (\exists\, y)(x,y) \in [\![\mathit{writtenBy}]\!]_A\} \quad \mathit{iff}\\
&\quad [\![\mathit{Paper}]\!]_A \subseteq \operatorname{dom} [\![\mathit{writtenBy}]\!]_A
\end{aligned}
$$

and

$$
\begin{array}{l}
A \models \top \sqsubseteq \forall\, writtenBy.Author \quad \textit{iff} \\
\llbracket \top \rrbracket_A \subseteq \llbracket \forall\, writtenBy.Author \rrbracket_A \quad \textit{iff} \\
\Delta_A \subseteq \{x \mid (\forall\, y)(x, y) \in \llbracket writtenBy \rrbracket_A \Rightarrow y \in \llbracket Author \rrbracket_A\} \\
\textit{iff} \\
(\forall\, x, y \in \Delta_A)(x, y) \in \llbracket writtenBy \rrbracket_A \Rightarrow y \in \llbracket Author \rrbracket_A \ \textit{iff} \\
\mathrm{ran}\, writtenBy \subseteq \llbracket Author \rrbracket_A
\end{array}
$$

Theorem 1 $\mathfrak{O} = (\mathsf{Sign}(\mathfrak{O}), \mathsf{sen}(\mathfrak{O}), \mathsf{Mod}(\mathfrak{O}), \models_{\mathfrak{O}})$, *where* $\models_{\mathfrak{O}}$ *associates with each OWL signature* $\mathcal{O}$ *the relation* $\models_{\mathcal{O}}$ *defined as above, is an institution.*

The next result proves the first main feature of the OWL institution.

Theorem 2 *The category of OWL signatures* $\mathsf{Sign}(\mathfrak{O})$ *is cocomplete.*

The proof of the next corollary follows from Theorem 27 in [31] and it supplies the mathematical support for putting together smaller ontologies to form larger ones.

Corollary 1 *The category* $\mathsf{Th}_{\mathfrak{O}}$ *is cocomplete.*

The second main feature of the OWL institution is given by the following result and it shows that there is a sound way to amalgamate consistent OWL models (worlds of resources).

Theorem 3 *The functor* $\mathsf{Mod}(\mathfrak{O}) : \mathsf{Sign}(\mathfrak{O})^{\mathrm{op}} \to \mathsf{Cat}$ *preserves pullbacks.*

5.1.1 The Grothendieck Institution of OWL

Since the institution we defined above is strongly dependent on the data type $(\mathbb{D}, [\![\_]\!])$, it follows that we should denote it by $\mathfrak{O}(\mathbb{D}, [\![\_]\!])$. The data type can be organized into a category DT as follows:

– the objects are pairs of the form $(\mathbb{D}, [\![\_]\!] : \mathbb{D} \to |\,\mathsf{Set}\,|)$

– the arrows $u : (\mathbb{D}, [\![\_]\!]) \to (\mathbb{D}', [\![\_]\!]')$ are functions $u : \mathbb{D} \to \mathbb{D}'$ such that $[\![D]\!] = [\![u(D)]\!]$ for all $D \in \mathbb{D}$.

We define the functor $\mathsf{owl} : \mathsf{DT}^{op} \to \mathsf{Ins}$ as follows:

– $\mathsf{owl}(\mathbb{D}, [\![\_]\!]) = \mathfrak{O}(\mathbb{D}, [\![\_]\!])$;

– if $u : (\mathbb{D}, [\![\_]\!]) \to (\mathbb{D}', [\![\_]\!]')$, then $\mathsf{owl}(u)$ is the institution morphism $(\phi^u, \alpha^u, \beta^u)$ where ϕ^u is the identity, $\alpha^u_{\mathcal{O}} : \mathsf{sen}(\mathfrak{O}(\mathbb{D}, [\![\_]\!]))(\mathcal{O}) \to \mathsf{sen}(\mathfrak{O}(\mathbb{D}', [\![\_]\!]'))(\mathcal{O})$ maps each $\mathcal{O}$-sentence F over $\mathbb{D}$ to an $\mathcal{O}$-sentence F' over $\mathbb{D}'$ obtained from F by replacing the occurrences of $D \in \mathbb{D}$ with $u(D)$, and $\beta^u_{\mathcal{O}}$ is identity.

The general institution of the web ontologies $\mathfrak{O}$ can now be defined as the Grothendieck institution $\mathsf{owl}^{\#}$.

The Grothendieck construction can be done in a more general framework. Let $\underline{\widehat{\mathrm{DT}}}$ be an institution of data types. The signature category of the predefined types is the Grothendieck category $\mathsf{Mod}(\underline{\widehat{\mathrm{DT}}})^{\#}$. The institution $\mathfrak{O}$ is now the Grothendieck institution of the indexed institution $\mathsf{owl} : (\mathsf{Mod}(\underline{\widehat{\mathrm{DT}}})^{\#})^{op} \to \mathsf{Ins}$. A main consequence of this fact is that we can change the syntactical notation for the data values or the implementation of the same abstract data type without changing the properties of the ontologies.

5.2 The Institution $\mathfrak{Z}$

Z [107, 89] is a formal specification language based on first-order predicate logic and ZF set theory. It is well suited for modeling system data and states. Z has a rich set of language constructs including given type, abbreviation type, axiomatic definition, state and operation schema definitions, etc.

We briefly recall from [3] the institution $\mathfrak{Z}$, denoted by $\mathcal{S}$ in [3], formalizing the logic underlying the specification language Z.

A *Z signature* $\mathcal{Z}$ is a triple (G, Op, τ) where G is the set of the *given-sets names*, Op is a set of the *identifiers*, and τ is a function mapping the names in Op into types $\mathcal{T}(G)$, where $\mathcal{T}(G)$ is inductively defined by:

1. $G \subseteq \mathcal{T}(G)$,
2. $T_1 \times \cdots \times T_n \in \mathcal{T}(G)$ for $T_i \in \mathcal{T}(G)$, $i = 1, \ldots, n$,
3. $\mathcal{P}(T) \in \mathcal{T}(G)$ for $T \in \mathcal{T}(G)$,
4. $\langle x_1 : T_1, \ldots, x_n : T_n \rangle \in \mathcal{T}(G)$ for $T_i \in \mathcal{T}(G)$ and x_i is a variable name, $i = 1, \ldots, n$, such that $i \neq j \Rightarrow x_i \neq x_j$.

A *Z signature morphism* $\phi : (G, Op, \tau) \to (G', Op', \tau')$ is a pair of functions $\phi_{gs} : G \to G'$ and $\phi_{op} : Op \to Op'$ such that $\tau \,;\mathcal{T}(\phi_{gs}) = \phi_{op} \,; \tau'$. $\mathcal{T}(\phi_{gs})$ is the standard extension of ϕ_{gs} to $\mathcal{T}(\phi_{gs}) : \mathcal{T}(G) \to \mathcal{T}(G')$. We denote by $\mathsf{Sign}(\mathfrak{Z})$ the category of Z signatures. Given a Z signature $\mathcal{Z} = (G, Op, \tau)$, a *$\mathcal{Z}$-structure (model)* is a pair (A_G, A_{Op}) where A_G is a functor from G, viewed as a discrete category, to Set, and A_{Op} is a set $\{(o, v) \mid o \in Op\}$ where $v \in \overline{A}_G(\tau(o))$. The functor $\overline{A}_G$ is the standard extension of A_G to $\overline{A}_G : \mathcal{T}(G) \to \mathsf{Set}$. A *$\mathcal{Z}$-homomorphism* $h : (A_G, A_{Op}) \to (B_G, B_{Op})$ is a natural transformation $h : A_G \Rightarrow B_G$ given by $\overline{h}_{\tau(o)}(v) = v'$, where $(o, v) \in A_{Op}$ and $(o, v') \in B_{Op}$; again, $\overline{h}$ is the usual extension of h to $\overline{h} : \overline{A}_G \Rightarrow \overline{B}_G$. We denote by $\mathsf{Mod}(\mathfrak{Z})(\mathcal{Z})$ the category of $\mathcal{Z}$-structures. Given a Z signature morphism

$\phi : \mathcal{Z} \to \mathcal{Z}'$ and a $\mathcal{Z}'$-structure $A' = (A'_{G'}, A'_{O'})$, the ϕ-reduct $A'\lceil_\phi$ is the $\mathcal{Z}$-structure $A = (A_G, A_{Op})$ given by $A_G = \phi_{gs} \, ; A'_{G'}$ and $A_{Op} = \{(o, v) \mid (\phi_{op}(o), v) \in A'_{Op'}, o \in Op\}$.

Given a Z signature $\mathcal{Z}$, the sets of *$\mathcal{Z}$-expressions* E, *$\mathcal{Z}$-schema-expressions* S, and (part) of *$\mathcal{Z}$-formulas* P are defined by:

$$
\begin{aligned}
E ::= \;& id \mid x \mid (E, \ldots, E) \mid E.i \mid \langle x_1 \mapsto E, \ldots, x_n \mapsto E \rangle \\
& \mid E.x \mid E(E) \mid \{E, \ldots, E\} \mid \{S \bullet E\} \mid \mathcal{P}(E) \\
& \mid E \times \cdots \times E \mid S \\
S ::= \;& x_1 : E;\ \ldots;\ x_n : E \mid (S \mid P) \mid \neg\, S \mid S \vee S \mid S \wedge S \\
& \mid S \Rightarrow S \mid \forall\, S.S \mid \exists\, S.S \mid S \setminus [x_1, \ldots, x_n] \\
& \mid S[x_1/y_1, \ldots, x_n/y_n] \mid S\ Decor \mid E \\
P ::= \;& \texttt{true} \mid \texttt{false} \mid E \in E \mid E = E \mid \neg\, P \mid P \vee P \mid P \wedge P \\
& P \Rightarrow P \mid \forall\, S.P \mid \exists\, S.P
\end{aligned}
$$

Example 3 *The following simple Z specification:*

$[Class, Resource]$

ClassesAsResources

$instances : Class \to \mathbb{P}\, Resource$
$res : Class \rightarrowtail Resource$

$\forall\, c, c' : Class;\ r : Resource;\ pr : \mathbb{P}\, Resource \bullet$
$\quad c \mapsto r \in res \Rightarrow \neg(r \in pr \wedge c' \mapsto pr \in instances)$

is described in the terms of the institution $\mathfrak{Z}$ *as* $\mathsf{CR} = ((G, Op, \tau), P)$ *where* $G = \{Class, Resource\}$, $Op = \{instances, res\}$, $\tau(instances) = \mathcal{P}(Class \times \mathcal{P}(Resource))$, $\tau(res) = \mathcal{P}(Class \times Resource)$, *and* P *includes the formulas expressing the functionality of the relation instances, the functionality and the injectivity of the relation res, together with the invariant of the state schema ClassesAsResources. It is easy to see, e.g., that* $c \mapsto r \in res$ *is a* CR*-expression and* $c, c' : Class;\ r : Resource;\ pr : \mathbb{P}\, Resource$ *is a* CR*-schema-expression.*

An *environment* $(\mathcal{Z}, (X, \tau_X))$ consists of a Z signature $\mathcal{Z}$, a set of variables $X = \{x_1, \ldots, x_n\}$, and a function $\tau_X : X \rightarrow \mathcal{T}(G)$ which associates a type with each variable. The sets of expressions and formulas are restricted to those well-formed w.r.t. an environment. Intuitively, an expression is *well-formed w.r.t. the environment* $(\mathcal{Z}, (X, \tau_X))$ iff we can uniquely associate to it a type which can be deduced from τ and τ_X. A $\mathcal{Z}$-formula P is *well defined w.r.t. the environment* $(\mathcal{Z}, (X, \tau_X))$ iff all its operators and quantifiers are given over expressions having the types compatible with their definition. For instance, if $X = \{c, r\}$, $\tau_X(c) = \mathit{Class}$, $\tau_X(r) = \mathit{Resource}$, then $c \mapsto r \in \mathit{res}$ is well defined w.r.t. the environment $(\mathsf{CR}, (X, \tau_X))$ whereas $c \mapsto r \in \mathit{instances}$ is not. Given a Z signature $\mathcal{Z}$ and an environment $(\mathcal{Z}, (X, \tau_X))$, a *variable binding* $\beta = (A, A_X)$ consists of a $\mathcal{Z}$-structure A and a set $A_X = \{(x_1, v_1), \ldots, (x_n, v_n)\}$ with $v_i \in \overline{A}_G(\tau_X(x_i))$ for $i = 1, \ldots, n$. The satisfaction relation between variable bindings and $\mathcal{Z}$-expressions and $\mathcal{Z}$-formulas is defined as expected (see [3] for details). For instance, if we consider the variable binding $\beta = (A, A_X)$, where $A_X = \{(c, v_c), (r, v_r)\}$, then $\beta \models c \mapsto r \in \mathit{res}$ iff $(v_c, v_r) \in w$ and $(\mathit{res}, w) \in A_{Op}$. The $\mathcal{Z}$*-sentences* are the $\mathcal{Z}$-formulas well defined with the environment $(\mathcal{Z}, (\{\,\}, \tau_{\{\,\}}))$. A $\mathcal{Z}$-structure A *satisfies* a $\mathcal{Z}$-sentence P, written $A \models_{\mathfrak{Z},\mathcal{Z}} P$, iff $(A, \{\,\}) \models P$.

The institution $\mathfrak{Z}$ is given by $\mathfrak{Z} = (\mathsf{Sign}(\mathfrak{Z}), \mathsf{sen}(\mathfrak{Z}), \mathsf{Mod}(\mathfrak{Z}), \models_{\mathfrak{Z}})$, where $\mathsf{Sign}(\mathfrak{Z})$ is the category of Z signatures, the functor $\mathsf{sen}(\mathfrak{Z})$ maps each Z signature $\mathcal{Z}$ to its set of $\mathcal{Z}$-sentences, the functor $\mathsf{Mod}(\mathfrak{Z})$ maps each Z signature $\mathcal{Z}$ to the category of $\mathcal{Z}$-structures, and $\models_{\mathfrak{Z},\mathcal{Z}}$ is defined as above.

5.2.1 The Use of the Mathematical Tool-kit

Many Z specifications use mathematical definitions included in so-called the Mathematical Tool-kit or standard library [89]. The use of these definitions can be formally described in terms of the structured specifications. We show that by means of an example. Let Z be the following Z specification:

$[\mathit{Resource}]$

$$\begin{array}{|l} \mathit{Class} : \mathbb{P}\,\mathit{Resource} \\ \mathit{Property} : \mathbb{P}\,\mathit{Resource} \\ \hline \mathit{Class} \cap \mathit{Property} = \emptyset \end{array}$$

In terms of the institution theory, the above specification is represented by $(\mathcal{Z}_0, P_0)$, where $G_0 = \{\text{Resource}\}$, $Op_0 = \{\text{Class}, \text{Property}\}$, $\tau_0(\text{Class}) = \tau_0(\text{Property}) = \mathcal{P}(\text{Resource})$, and $P_0 = \{\text{Class} \cap \text{Property} = \emptyset\}$. The definitions for $\emptyset$, meaning $\emptyset[\text{Resource}]$, and $\_\cap\_$ are included in the standard library:

$\emptyset[X] ::= \{x : X \mid \mathit{false}\}$

$$\begin{array}{|l} [X] \\ \hline \_\cap\_ : \mathbb{P}\,X \times \mathbb{P}\,X \rightarrow \mathbb{P}\,X \\ \hline \forall S, T : \mathbb{P}\,X \bullet S \cap T = \{x : X \mid x \in S \wedge x \in T\} \\ \hline \end{array}$$

The full description $(\mathcal{Z}, P)$ of the initial Z specification is obtained as the vertex of the following pushout:

$$\begin{array}{ccc} \emptyset[\text{Resource}] & \longrightarrow & \_\cap\_[\text{Resource}] \\ \downarrow & & \downarrow \\ (\mathcal{Z}_0, P_0) & \longrightarrow & (\mathcal{Z}, P) \end{array}$$

5.3 Encoding $\mathfrak{O}$ in $\mathfrak{Z}$

In previous two chapters, we developed the semantics for DAML+OIL language in formal language Z as a extension of the standard library. This semantic library was later on revised for the new ontology language OWL, incorporating changes incurred in OWL from DAML+OIL. In this Section, we will demonstrate, through institutions comorphisms, that the Z encoding of OWL is indeed sound.

The main idea is to associate a Z specification $\Phi(\mathcal{O}, F)$ with each OWL specification $(\mathcal{O}, F)$ such that an $(\mathcal{O}, F)$-model can be extracted from each $\Phi(\mathcal{O}, F)$-model. The construction of $\Phi(\mathcal{O}, F)$ is given in two steps: we first associate a Z specification $\Phi(\mathcal{O})$ with each OWL signature $\mathcal{O}$ and then we add to it the sentences F translated via a natural transformation.

Since $\Phi(\mathcal{O}, F)$ can be seen as a Z semantics of $(\mathcal{O}, F)$, it includes a distinct subspecification $(\mathcal{Z}^{\emptyset}, P^{\emptyset})$ defining the main OWL concepts and the operations over sets. More precisely, we consider $(\mathcal{Z}^{\emptyset}, P^{\emptyset})$ as being the vertex of the colimit having as base the standard library, the specification of the data types, together with the following Z specification:

given sets:
`Resource`;

identifiers:

- √ corresponding to OWL signatures:
  `Class`, `Property`, `ObjectProperty`, `DatatypeProperty`,
  `Individual`, `Thing`, `Nothing`
- √ giving Z semantics to OWL signatures:
  `instances`, `subVal`
- √ corresponding to OWL class axioms:
  `disjointClasses`, `equivalentClasses`, `subClassOf`
- √ corresponding to OWL descriptions and restrictions:
  `unionOf`, `intersectionOf`, `complementOf`, `oneOf`,
  `allValuesFrom`, `someValuesFrom`,
  `minCardinality`, `maxCardinality`, `cardinality`
- √ corresponding to OWL property axioms:
  `domain`, `range`, `functional`, `inverseOf`, `symmetric`,
  `transitive`, `inverseFunctional`,
  `equivalentProperties`, `subPropertyOf`

$\tau^{\emptyset}$ **for the new identifiers:**

- $\surd$ corresponding to OWL signatures:
  $\tau^{\emptyset}(\mathtt{Class}) = \tau^{\emptyset}(\mathtt{Property}) = \tau^{\emptyset}(\mathtt{ObjectProperty}) =$
  $\tau^{\emptyset}(\mathtt{DatatypeProperty}) = \mathcal{P}(\mathtt{Resource})$
  $\tau^{\emptyset}(\mathtt{Thing}) = \tau^{\emptyset}(\mathtt{Nothing}) = \mathtt{Resource}$
- $\surd$ giving Z semantics to OWL signatures:
  $\tau^{\emptyset}(\mathtt{instances}) = \mathcal{P}(\mathtt{Resource} \times \mathcal{P}(\mathtt{Resource}))$
  $\tau^{\emptyset}(\mathtt{subVal}) = \mathcal{P}(\mathtt{Resource} \times \mathcal{P}(\mathtt{Resource} \times \mathtt{Resource}))$
- $\surd$ corresponding to OWL class axioms:
  $\tau^{\emptyset}(\mathtt{disjointClasses}) = \tau^{\emptyset}(\mathtt{Class} \times \mathtt{Class})$
  $= \mathcal{P}(\mathtt{Resource} \times \mathtt{Resource})$
  $\tau^{\emptyset}(\mathtt{equivalentClasses}) = \mathcal{P}(\mathtt{Resource} \times \mathtt{Resource})$
  $\tau^{\emptyset}(\mathtt{subClassOf}) = \mathcal{P}(\mathtt{Resource} \times \mathtt{Resource})$
- $\surd$ corresponding to OWL descriptions, restrictions
  $\tau^{\emptyset}(\mathtt{unionOf}) = \tau^{\emptyset}((\mathtt{Class} \times \mathtt{Class}) \times \mathtt{Class})$
  $= \mathcal{P}((\mathtt{Resource} \times \mathtt{Resource}) \times \mathtt{Resource})$
  $\tau^{\emptyset}(\mathtt{intersectionOf}) = \mathcal{P}((\mathtt{Resource} \times \mathtt{Resource}) \times \mathtt{Resource})$
  $\tau^{\emptyset}(\mathtt{complementOf}) = \mathcal{P}(\mathtt{Resource} \times \mathtt{Resource})$
  $\tau^{\emptyset}(\mathtt{oneOf}) = \mathcal{P}(\mathcal{P}(\mathtt{Resource}) \times \mathtt{Resource})$
  $\tau^{\emptyset}(\mathtt{allValuesFrom}) = \tau^{\emptyset}((\mathtt{Resource} \times \mathtt{Property}) \times \mathtt{Class})$
  $= \mathcal{P}((\mathtt{Resource} \times \mathtt{Resource}) \times \mathtt{Resource})$
  ...
- $\surd$ corresponding to OWL property axioms:
  $\tau^{\emptyset}(\mathtt{domain}) = \tau^{\emptyset}(\mathtt{Property} \times \mathtt{Resource})$
  $= \mathcal{P}(\mathtt{Resource} \times \mathtt{Resource})$
  $\tau^{\emptyset}(\mathtt{range}) = \mathcal{P}(\mathtt{Resource} \times \mathtt{Resource})$
  $\tau^{\emptyset}(\mathtt{inverseOf}) = \tau^{\emptyset}(\mathtt{ObjectProperty} \times \mathtt{ObjectProperty})$
  $= \mathcal{P}(\mathtt{Resource} \times \mathtt{Resource})$
  $\tau^{\emptyset}(\mathtt{functional}) = \tau^{\emptyset}(\mathtt{Property}) = \mathcal{P}(\mathtt{Resource})$
  ...

sentences :

- $\surd$ corresponding to OWL signatures:
  $\mathtt{Class} \cap \mathtt{Property} = \emptyset$
  $\mathtt{Class} \cap \mathtt{Individual} = \emptyset$
  $\mathtt{Property} \cap \mathtt{Individual} = \emptyset$
  $\mathtt{ObjectProperty} \cap \mathtt{DatatypeProperty} = \emptyset$
  $\mathtt{Property} = \mathtt{ObjectProperty} \cup \mathtt{DatatypeProperty}$
- $\surd$ giving Z semantics to OWL signatures:
  $\mathtt{instances}(\mathtt{Thing}) = \mathtt{Individual}$
  $\mathtt{instances}(\mathtt{Nothing}) = \emptyset$
  $\forall\, c : \mathtt{Class} \bullet \mathtt{instances}(c) \subseteq \mathtt{Individual}$

$\forall p : \texttt{Property} \bullet \texttt{subVal}(p) \subseteq \mathcal{P}(\texttt{Individual} \times \texttt{Resource})$
$\ldots$

$\surd$ corresponding to OWL class axioms:

$\forall c_1, c_2 : \texttt{Class} \bullet c_1 \mapsto c_2 \in \texttt{disjointClasses} \Leftrightarrow$
$\quad \texttt{instances}(c_1) \cap \texttt{instances}(c_2) = \emptyset$
$\forall c_1, c_2 : \texttt{Class} \bullet c_1 \mapsto c_2 \in \texttt{subClassOf} \Leftrightarrow$
$\quad \texttt{instances}(c_1) \subseteq \texttt{instances}(c_2)$
$\forall c_1, c_2 : \texttt{Class} \bullet c_1 \mapsto c_2 \in \texttt{equivalentClasses} \Leftrightarrow$
$\quad \texttt{instances}(c_1) = \texttt{instances}(c_2)$

$\surd$ corresponding to OWL descriptions, restrictions:

$\forall c, c_1, c_2 : \texttt{Class} \bullet (c_1, c_2) \mapsto c \in \texttt{unionOf} \Leftrightarrow$
$\quad \texttt{instances}(c) = \texttt{instances}(c_1) \cup \texttt{instances}(c_2)$
$\forall p : \texttt{Property};\ c_1, c : \texttt{Class} \bullet$
$\quad (p, c_1) \mapsto c \in \texttt{allValuesFrom} \Leftrightarrow$
$\quad \texttt{instances}(c) = \{x : \texttt{Individual} \mid \forall y : \texttt{Individual} \bullet$
$\qquad (x, y) \in \texttt{subVal}(p) \Rightarrow y \in \texttt{instances}(c_1)\}$
$\forall p : \texttt{Property};\ n : \mathbb{N};\ c : \texttt{Class} \bullet$
$\quad (p, n) \mapsto c \in \texttt{minCardinality} \Leftrightarrow$
$\quad \texttt{instances}(c) =$
$\quad \{x : \texttt{Individual} \mid \#(\texttt{subVal}(p)(\!|\ \{x\}\ |\!)) \leq n\}$
$\ldots$

$\surd$ corresponding to OWL property axioms:

$\forall p_1, p_2 : \texttt{Property} \bullet p_1 \mapsto p_2 \in \texttt{subPropertyOf} \Leftrightarrow$
$\quad \texttt{subVal}(p_1) \subseteq \texttt{subVal}(p_2)$
$\forall p : \texttt{Property};\ c : \texttt{Class} \bullet p \mapsto c \in \texttt{domain} \Leftrightarrow$
$\quad \operatorname{dom} \texttt{subVal}(p) \subseteq \texttt{instances}(c)$
$\forall p : \texttt{Property} \bullet p \in \texttt{functional} \Leftrightarrow$
$\quad \forall x, y, z : \texttt{Resource}(x, y) \in \texttt{subVal}(p) \wedge$
$\qquad (x, z) \in \texttt{subVal}(p) \Rightarrow y = z$
$\ldots$

We define $\Phi^\diamond : \mathsf{Sign}(\mathfrak{O}) \to \mathsf{Sign}(\mathfrak{Z})$ as follows. Let $\mathcal{O} = (\mathbb{C}, \mathbb{R}, \mathbb{U}, \mathbb{I})$ be an OWL signature. Then $\Phi^\diamond(\mathcal{O}) = (G, Op, \tau)$ is defined as follows:

$G = G^{\emptyset}$;

$Op = Op^{\emptyset} \cup \mathbb{C} \cup \mathbb{R} \cup \mathbb{U} \cup \mathbb{I}$;

$\tau(C) = \texttt{Resource}$ for each $C \in \mathbb{C}$,

$\tau(R) = \texttt{Resource}$ for each $R \in \mathbb{R}$,

$\tau(U) = \texttt{Resource}$ for each $U \in \mathbb{U}$,

$\tau(o) = \mathtt{Resource}$ for each $o \in \mathbb{I}$.

If $\varphi : \mathcal{O} \to \mathcal{O}'$ is an OWL signature morphism and $\Phi^\diamond(\mathcal{O}) = (G^\emptyset, Op, \tau)$ and $\Phi^\diamond(\mathcal{O}') = (G^\emptyset, Op', \tau')$, then $\Phi^\diamond(\varphi) : \Phi(\mathcal{O}) \to \Phi(\mathcal{O}')$ is the Z signature morphism $(\mathrm{id} : G^\emptyset \to G^\emptyset, \Phi^\diamond(\varphi)_{op} : Op \to Op')$ such that $\Phi^\diamond(\varphi)_{Op}$ is the identity over the subset $Op^\emptyset$ and $\Phi^\diamond(\varphi)_{op}(N) = \varphi(N)$ for each name N in $\mathcal{O}$. It is easy to check that $\tau\,;\mathcal{T}(\mathrm{id}) = \Phi^\diamond(\varphi)_{op}\,;\tau'$.

We extend $\Phi^\diamond$ to $\Phi : \mathsf{Sign}(\mathfrak{O}) \to \mathsf{Th}(\mathfrak{Z})$ by defining $\Phi(\mathcal{O}) = (\Phi^\diamond(\mathcal{O}), P)$, where P is $P^\emptyset$ together with the following sentences:

$$\{C \in \mathtt{Class}) \mid C \in \mathbb{C}\} \cup$$
$$\{R \in \mathtt{ObjectProperty} \mid R \in \mathbb{R}\} \cup$$
$$\{U \in \mathtt{DatatypeProperty} \mid U \in \mathbb{U}\} \cup$$
$$\{o \in \mathtt{Individual} \mid o \in \mathbb{I}\}.$$

If $\mathcal{O}$ is an OWL signature, then

$$\alpha_{\mathcal{O}} : \mathsf{sen}(\mathfrak{O})(\mathcal{O}) \to \mathsf{sen}(\mathfrak{Z})(\Phi(\mathcal{O}))$$

is defined by:

$$\alpha_{\mathcal{O}}(\bot) = \mathtt{Nothing},\ \alpha_{\mathcal{O}}(\top) = \mathtt{Thing},$$
$$\alpha_{\mathcal{O}}(N) = N \text{ for each name } N \text{ in } \mathcal{O}$$
$$\alpha_{\mathcal{O}}(C_1 \sqcap C_2) = \mathtt{intersectionOf}(\alpha_{\mathcal{O}}(C_1), \alpha_{\mathcal{O}}(C_2)),$$
$$\ldots$$
$$\alpha_{\mathcal{O}}(\forall\, R.C) = \mathtt{allValuesFrom}(\alpha_{\mathcal{O}}(R), \alpha_{\mathcal{O}}(C)),$$
$$\ldots$$
$$\alpha_{\mathcal{O}}(\leqslant n\, R) = \mathtt{maxCardinality}(\alpha_{\mathcal{O}}(R), n),\ \ldots$$
$$\alpha_{\mathcal{O}}(C_1 \sqsubseteq C_2) = \alpha_{\mathcal{O}}(C_1) \mapsto \alpha_{\mathcal{O}}(C_2) \in \mathtt{subClassOf},$$
$$\ldots$$
$$\alpha_{\mathcal{O}}(E) = \{\alpha_{\mathcal{O}}(e) \mid e \in E\}.$$

Example 4 *Let $\mathcal{O}$ be that defined in Example 1.*

$\alpha_{\mathcal{O}}(Paper \sqsubseteq \ \geqslant 1\, writtenBy) =$
$Paper \mapsto \mathtt{minCardinality}(writtenBy, 1) \in \mathtt{subclassOf}$
which is equivalent to
$\mathtt{instances}(Paper) \subseteq \mathrm{dom}\ \mathtt{subVal}(writtenBy)$

$\alpha_{\mathcal{O}}(\top \sqsubseteq \forall\, writtenBy.Author) =$
$\mathtt{Resource} \mapsto \mathtt{allValuesFrom}(\mathtt{Author}, \mathtt{writtenBy})$
$\quad \in \mathtt{subclassOf}$
which is equivalent to
$\mathrm{ran}\ \mathtt{subVal}(writtenBy) \subseteq \mathtt{instances}(Author)$

Lemma 1 $\alpha = \{\alpha_{\mathcal{O}} \mid \mathcal{O} \in \mathsf{Sign}(\mathfrak{O})\}$ *is a natural transformation* $\alpha : \mathsf{sen}(\mathfrak{O}) \Rightarrow \Phi^{\diamond}; \mathsf{sen}(\mathfrak{Z})$.

Proof: Let $\varphi : \mathcal{O} \to \mathcal{O}'$ be an OWL signature morphism. Then it is a matter of routine to check that the following diagram commutes:

$$\begin{array}{ccc} \mathsf{sen}(\mathfrak{O})(\mathcal{O}) & \xrightarrow{\alpha_{\mathcal{O}}} & \mathsf{sen}(\mathfrak{Z})(\Phi^{\diamond}(\mathcal{O})) \\ {\scriptstyle \mathsf{sen}(\mathfrak{O})(\varphi)}\downarrow & & \downarrow{\scriptstyle \mathsf{sen}(\mathfrak{Z})(\Phi(\varphi))} \\ \mathsf{sen}(\mathfrak{O})(\mathcal{O}') & \xrightarrow[\alpha_{\mathcal{O}'}]{} & \mathsf{sen}(\mathfrak{Z})(\Phi^{\diamond}(\mathcal{O}')) \end{array}$$

For instance, if $C_1, C_2 \in \mathbb{C}$, then $\alpha_{\mathcal{O}}(C_1 \sqsubseteq C_2) = (C_1 \mapsto C_2 \in \mathtt{subClsassOf})$ and $\mathsf{sen}(\mathfrak{Z})(\Phi^{\diamond}(\phi))(\alpha_{\mathcal{O}}(C_1 \sqsubseteq C_2)) = (\phi(C_1) \mapsto \phi(C_2) \in \mathtt{subClsassOf})$. On the other hand, $\mathsf{sen}(\mathfrak{O})(\phi)(C_1 \sqsubseteq C_2) = (\phi(C_1) \sqsubseteq \phi(C_2))$ and $\alpha_{\mathcal{O}}(\phi(C_1) \sqsubseteq \phi(C_2)) = (\phi(C_1) \mapsto \phi(C_2) \in \mathtt{subClsassOf})$. □

If $\mathcal{O} = (\mathbb{C}, \mathbb{R}, \mathbb{U}, \mathbb{I})$ is an OWL signature and $A' = (A'_G, A'_{Op})$ a $\Phi^{\diamond}(\mathcal{O})$-model, then $\beta_{\mathcal{O}}(A')$ is the $\mathcal{O}$-model $A = (\Delta_A, [\![\_]\!]_A, Res_A, res_A)$ defined as follows:

$Res_A = A'_G(\mathtt{Resource})$,
$res_A(N) = v$ where $(N, v) \in A'_{Op}$ for each name $N \in \mathcal{O}$,
$\Delta_A = v$ where $(\mathtt{Thing}, v) \in A'_{Op}$,
if $C \in \mathbb{C}$, then $[\![C]\!]_A = v_C$ where $(\mathtt{instances}, v) \in A'_{Op}$ and $(C, v_C) \in v$,

if $R \in \mathbb{R}$, then $[\![R]\!]_A = v_R$ where $(\mathtt{subVal}, v) \in A'_{Op}$ and $(R, v_R) \in v$,
if $U \in \mathbb{U}$, then $[\![U]\!]_A = v_U$ where $(\mathtt{subDVal}, v) \in A'_{Op}$ and $(U, v_U) \in v$.

A is indeed an $\mathcal{O}$-model. For instance, if $(\mathtt{instances}, v) \in A'_{Op}$, then v is the graph of the function defined in A' by $\mathtt{instances}$ and v_C is just the value of this function for the argument C. Since $\tau^{\emptyset}(\mathtt{instances}) = \mathcal{P}(\mathtt{Resource} \times \mathcal{P}(\mathtt{Resource}))$, it follows that $v_C \subseteq A'_G(\mathtt{Resource})$. We obtain $[\![C]\!]_A \subseteq \Delta_A$ applying the sentences in $P^{\emptyset}$. We extend $\beta_{\mathcal{O}}$ to a functor $\beta_{\mathcal{O}} : \mathsf{Mod}'(\Phi^{\diamond}(\mathcal{O})) \to \mathsf{Mod}(\mathcal{O})$ as follows: if $h : A' \to B'$ is a $\Phi^{\diamond}(\mathcal{O})$-homomorphism, then $\beta_{\mathcal{O}}(h)$ is the $\mathcal{O}$-homomorphism $\beta_{\mathcal{O}}(h) : \beta_{\mathcal{O}}(A') \to \beta_{\mathcal{O}}(B')$ given by $\beta_{\mathcal{O}}(h) = h_{\mathtt{Resource}}$.

Lemma 2 $\beta = \{\beta_{\mathcal{O}} \mid \mathcal{O} \in \mathsf{Sign}(\mathfrak{O})\}$ *is a natural transformation* $\beta : \Phi^{\diamond\, op}; \mathsf{Mod}(\mathfrak{Z}) \Rightarrow \mathsf{Mod}(\mathfrak{O})$.

Proof: Let $\varphi : \mathcal{O} \to \mathcal{O}'$ be an OWL signature morphism. The commutativity of the diagram:

$$\begin{array}{ccc} \mathsf{Mod}(\mathfrak{Z})(\Phi^{\diamond\, op}(\mathcal{O}')) & \xrightarrow{\beta_{\mathcal{O}'}} & \mathsf{Mod}(\mathfrak{O})(\mathcal{O}') \\ {\scriptstyle \mathsf{Mod}(\mathfrak{Z})(\Phi^{\diamond\, op}(\varphi))}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{Mod}(\mathfrak{O})(\varphi^{op})} \\ \mathsf{Mod}(\mathfrak{Z})(\Phi^{\diamond\, op}(\mathcal{O})) & \xrightarrow{\beta_{\mathcal{O}}} & \mathsf{Mod}(\mathfrak{O})(\mathcal{O}) \end{array}$$

follows by checking that $\beta_{\mathcal{O}}(A'\restriction_{\Phi^{\diamond\, op}(\varphi)}) = \beta_{\mathcal{O}'}(A')\restriction_{\varphi}$ for each $\Phi^{\diamond}(\mathcal{O})$-model A'. $\square$

Theorem 4 $(\Phi, \alpha, \beta) : \mathfrak{O} \to \mathfrak{Z}$ *is a simple theoroidal comorphism.*

Proof: We already proved that α and β are natural transformations. We have to prove the satisfaction condition. Let $\mathcal{O}$ be an OWL signature, e an $\mathcal{O}$-sentence, and A' a $\mathsf{Mod}(\mathfrak{Z})(\Phi(\mathcal{O}))$-model. We suppose first that $A' \models_{\Phi(\Sigma)} \alpha_{\mathcal{O}}(e)$. We prove that $\beta_{\mathcal{O}}(A') \models_{\mathcal{O}} e$ by structural induction on e. For instance, we suppose that e is $C_1 \sqsubseteq C_2$. We have:

$A' \models_{\Phi(\mathcal{O})} \alpha_{\mathcal{O}}(C_1 \sqsubseteq C_2)$ iff $A' \models_{\Phi(\mathcal{O})} C_1 \mapsto C_2 \in \texttt{subClassOf}$

Since $A' \models P^{\emptyset}$ (we recall that $\Phi(\mathcal{O}) = (\Phi^{\diamond}(\mathcal{O}), P^{\emptyset})$), it follows that $A' \models \forall\, c_1, c_2 : \texttt{Class} \bullet c_1 \mapsto c_2 \in \texttt{subClassOf} \Rightarrow \texttt{instances}(c_1) \subseteq \texttt{instances}(c_2))$ which implies $[\![C_1]\!]_{\beta_{\mathcal{O}}(A')} \subseteq [\![C_2]\!]_{\beta_{\mathcal{O}}(A')}$, i.e., $\beta_{\mathcal{O}}(A') \models C_1 \sqsubseteq C_2$. The inverse implication is proved in a similar way. □

5.4 Chapter Summary

The main contribution of this chapter is the formal proof of the soundness of the Z semantics of ontology language OWL DL, which is the semantical foundation of the combined approach presented in the previous chapter.

As ontology languages and Z (and Alloy) are based on different logical systems (description logics vs first-order predicate logic), the proof of semantical equivalence between the OWL language constructs and Z semantics has to resort to a higher-level device that is able to reason with different logical systems.

In this chapter, we used the notion of institutions and institution comorphisms to represent the two logical systems underlying OWL DL and Z. Two institutions, $\mathfrak{O}$ (for OWL DL) and $\mathfrak{Z}$ (for Z) were defined and we proved that there is a simple theoroidal comorphism $(\Phi, \alpha, \beta) : \mathfrak{O} \rightarrow \mathfrak{Z}$ between $\mathfrak{O}$ and $\mathfrak{Z}$. Hence, we proved the soundness of the Z semantics for OWL DL.